Analysis of progressive Type-II censoring in presence of competing risk data under step stress modeling

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Abstract

In this article we consider analysis of progressive Type-II censoring scheme in presence of competing risks under simple step stress modeling. We assume there are two competing risk factors in each stress level and lifetime distribution of each one of them is one parameter exponential distribution. Further, we assume the data set is following cumulative exposure model (CEM) in the stress levels. Based on the joint likelihood of the parameters, the conditional maximum likelihood estimators (MLEs) of the parameters are derived. Confidence intervals of the parameters based on their conditional MLEs are constructed. We also construct percentile bootstrap confidence intervals of the parameters. Further we carry out Bayesian analysis using Beta-gamma prior distribution to construct credible intervals of the parameters. A simulation experiment has been performed to observe the performances of the model. Finally one specific simulated data set has been analysed for illustrative purpose.

Key Words and Phrases: Competing risk; progressive Type-II censoring; competing risk; beta-gamma distribution; maximum likelihood estimator; bootstrap confidence interval; Bayes credible interval.

AMS Subject Classifications: 62F10, 62F03, 62H12.

1 Introduction

Censoring has become a part of life testing experiments where, the experiment terminates before all of the undergoing units fail. This is done to save and reduce duration of the experiment and cost incurred in manufacturing the units. There are several types of censoring schemes applied in a life testing experiment. Among them, Type-I and Type-II are the most popular censoring schemes. However in reality, it is not uncommon that some of the units break down accidentally during the course of the experiment. Unfortunately neither of Type-I or Type-II censoring scheme does allow to remove units during the experiment. Progressive Type-II censoring, introduced by Herd [27] is one of the popular schemes, where one can withdraw experimental units during the experiment. It can be described as follows. Suppose there are n items put on the test with prefixed integer $m(\leq n)$. Let us further choose non negative integers $R_1, R_2, \ldots, R_m$ such that, $m + \sum_{i=1}^{m} R_i = n$. At the failure time of i- th unit, say t_i, R_i number of units are withdrawn from the system. This procedure will continue till the failure of m- th unit from the system takes place with the remaining R_m number of units are removed from the system. It is to be noted that Type-II censoring scheme can be obtained by taking $R_1 = R_2 = \ldots = R_{m-1} = 0$ and $R_m = n - m$.

In a typical life testing experiment it may be difficult to get sufficient number of failures under normal operating conditions. This is because of the advancement of science and technology the products have become more long lasting and durable. In reality, it is very much possible that there is a sudden change in the environment which causes a working product to fail. For example, changes in environment including temperature changes, air pressure fluctuations, humidity variations etc. could be responsible to stop working an electronic equipment. To carry out a statistical analysis, the experimenter artificiality changes the environment of the undergoing experiment to get early failures. In statistical literature this is called as 'accelerated life testing' (ALT). There are different ways of conducting an ALT experiment. One of the popular methods is known as 'step stress life test' (SSLT). In an SSLT, the experimenter enforces different stress levels at different time points in the middle of the experiment. More specifically, in SSLT experiment suppose n units are placed in the experiment under the first or initial stress level S_1. The units are then subjected to $k-1$ prefixed number of different stress levels say, $S_2, \ldots, S_k$ at predefined time points $\tau_1 < \tau_2 < \ldots < \tau_{k-1}$ respectively. Suppose for $i = 1, 2, \ldots, k$, there are n_i units fail in the i-th stress level and $\sum_{j=1}^{k} n_j$. Then the data set is

$$\begin{aligned} data =& \Big\{ 0 < t_{1:n} < \ldots < t_{n_1:n} \leq \tau_1 < t_{n_1+1:n} < \ldots < t_{\sum_{j=1}^{2} n_j:n} \leq \tau_2 < t_{\sum_{j=1}^{2} n_j+1:n} < \ldots < \\ & < t_{\sum_{j=1}^{k-1} n_j:n} \leq \tau_{k-1} < t_{\sum_{j=1}^{k-1} n_j+1:n} \cdots < t_{\sum_{j=1}^{k} n_j:n} \Big\}. \end{aligned}$$

It is to be noted that although the data representation in the above is given for full sample data for SSLT, the different types of censoring schemes as been discussed before can be applied in the experiment. An SSLT is called simple SSLT if $k = 2$ *i.e.* there are two stress levels in the system.

There are different types of models assumptions used in SSLT experiment. The most common and popular model is cumulative exposure model (CEM). The CEM was first proposed by Seydyakin [41] and later studied by Bagdonavicius [3] and Nelson [38]. Under the assumptions of CEM, the residual life of a unit at a stress level depends only on the cumulative exposure that the unit has experienced, no matter how this exposure was accumulated. Let us consider a k stress SSLT experiment as described before with cumulative distribution function(CDF) of the units at i-th stress level being $F_i(.)$. Suppose $F(.)$ denotes the CDF of the units under CEM. Then under the assumptions of CEM, $F(.)$ is related to CDF under each stress level by the following set equations.

$$F(t) = F_i(t - h_{i-1}), \qquad \text{if } \tau_{i-1} < t < \tau_i, \quad i = 1, 2, \ldots, k, \tag{1}$$

with $\tau_0 = 0, \tau_k = \infty, h_0 = 0$ and h_i, for $i = 1, 2, \ldots, k-1$, is solution of the equation $F_i(\tau_i - h_{i-1}) = F_{i+1}(\tau_i - h_i)$.

In a reliability experiment, it is interesting to asses some specific risk factors in presence of other risk factors. In statistical literature this called competing risk problem. In a typical competing risk problem a unit may fail due to several causes. For example a working computer can all of a sudden shuts down due to any one or more of the reasons *viz.* failure of motherboard of the computer or the failure of central processing unit of the system or failure of may be some different reasons. In a typical competing risk experiment, one observes time of failure and the associated cause responsible for the failure of the units. Let δ be an indicator variable showing the cause of failure of a unit. Then the data set is

$$data = \big\{(z_{1:n}, \delta_1), (z_{2:n}, \delta_2), \ldots, (z_{n:n}, \delta_n)\big\}.$$

Here, $z_{i:n}$ denotes the failure time of i-th unit. Sometimes the exact cause of a failure may not be known. Hence a suitable analytical treatment is required to take care in those cases. Cox [18] suggested latent failure time models in which the risks are assumed to be independent to each other. In this model, suppose n units are put in the experiment and $X_1, X_2, \ldots, X_p$ are the random variables denoting the lifetimes of p competing risks responsible for the failure of units. Then a unit fails as soon as one of the competing risks causes the unit fail. Several works have been done under this assumption. Tsiatis [42] discussed the non identifiability aspect in competing risk model. He showed that given a joint survivor density function of different dependent causes, one can have independent causes having the same joint survivor function. Thus from the data set consisting of failure time points and causes of failures, it is not possible to identify whether the causes are independent or not.

Several works have been done in different life testing models using Progressive Type-II censoring scheme. One can see Balakrishnan and Aggarwala [5] for a book length reference and an excellent review on Progressive censoring by Balakrishnan [4] and the references therein of the developments on progressive Type-II censoring. Balakrishnan and Sandhu [13] considered maximum likelihood estimation and best linear unbiased estimation of the parameters in both one and two parameter exponential distribution for Progressive Type-II censoring. They showed both the estimators are same after some bias adjustment. Some of the other distributions used in Progressive Type-II censoring are logistic distribution by Balakrishnan and Kannan [9], laplace distribution by Aggarwala and Balakrishnan [1], half logistic distribution by Balakrishnan and Asgharzadeh [6], Gaussian distribution by Balakrishnan *et al.* [10] etc. Balakrishnan *et al.* [11] considered extreme value distribution in Progressive Type-II censoring. The MLEs of the parameters are not obtained in explicit form and hence approximate maximum likelihood estimates are computed.

Step stress modeling has been extensively studied by different authors. Balakrishnan *et al.* [16] and Balakrishnan *et al.* [12] considered simple SSLT with one parameter exponential model under Type-I and Type-II censoring scheme respectively. They found MLEs of the parameters and established exact conditional distributions of the parameters. Balakrishnan and Xie [14], [15] considered the exact inference for a simple step stress model with Type-I and Type-II Hybrid censoring. Later Balakrishnan and Xie [14] and Balakrishnan and Xie [15] derived exact inferential results for SSLT with one parameter exponential distribution under Type-I Hybrid and Type-II Hybrid censoring respectively. Mitra *et al.* [37] considered simple step stress model for two parameter exponential distribution under Type-II censoring. Gouno *et al.* [23] studied optimal choice of stress changing time points for both simple SSLT and multiple SSLT case with one parameter exponential distributions. They considered the mean life time to be a log linear function of stress levels. However there were some erroneous expressions in that work which were corrected by Han *et al.* [25]. Xie *et al.* [43] derived exact distributions of MLEs of the parameters of exponential distribution under Progressive Type-II censoring.

The competing risk modeling has been studied extensively in different distributions under different types of censoring schemes as well as along with ALT modeling. One can see Crowder [19], Crowder [20], David and Moeschberger [21] for book length references. Kundu and Gupta [32] considered competing risk modeling with one parameter exponential distribution with Type-I Hybrid censoring scheme. They considered both frequentist and Bayesian analysis of the data. Kundu *et al.* [34] and Kundu and Joarder [33] considered analysis of Progressive and Progressive Hybrid Type-I censored competing risk, respectively for exponential distributions. Ashour and Abu-El-Azm [2] analyzed Progressive Hybrid Type-II censored competing risk data with exponential distributions. They however did not establish exact distributions of the MLEs of the parameters. They carried out Bayesian analysis using LINEX loss function with gamma as prior distribution. Competing risk analysis for two parameter exponential distribution in Type-II Hybrid censoring has been discussed by Mao *et al.* [36]. Kundu and Basu [31] considered competing risk for incomplete data with exponential as well as Weibull populations. Bhattacharya *et al.* [17] analyzed Hybrid Type-II censored competing risk data with frequentist and Bayesian approaches with Weibull distributions. Since MLEs of the unknown parameters were not obtained in explicit form they used approximate maximum likelihood estimator to obtain MLEs in explicit form. Beta-Gamma distribution and a log concave density were used as prior distributions for Bayesian analysis. Pareek *et al.* [39] analyzed the competing risk data coming from two parameter Weibull distributions under progressive censoring. They obtained some optimal censoring schemes of the experiment design and proposed a sub optimal censoring scheme due to expensive computations in the former case. Bayesian analysis of competing risk data from Weibull distributions under Progressive Type-II censoring by Kundu and Pradhan [35]. Klein and Basu ([28], [29], [30]) used the method of maximum likelihood to estimate the model parameters for independent exponential or Weibull distributions under different censoring schemes *viz.* Type-I, Type-II, and progressively censored ALT observations. Han and Balakrishnan [24] considered simple step stress modeling in Type-I censoring scheme under competing risk set up with exponential distributions as the distribution of risk factors. Balakrishnan and Han [8] also considered in the same model with Type-II censoring instead of Type-I censoring scheme. In both the cases they established exact distributions of MLEs of the parameters. Han and Kundu [26] considered step stress modeling for Type-I censoring with generalized exponential risk factors. Ganguly and Kundu [22] analyzed competing risk in simple step stress model where the stress changing time is random with Type-II censoring. They established exact distributions of the MLEs of the parameters and studied some optimal tests for the choice of experiment design. In this article we have considered simple step stress modeling under progressive Type-II censoring in presence of competing risks. We have assumed CEM in the two stress levels. In first stress level the two distributions of two risk factors are assumed to be exponential distributions with mean θ_{11} and θ_{21}, respectively whereas in the second stress level the corresponding distributional assumptions become exponential with corresponding means θ_{12} and θ_{22}, respectively. In Section 2 we describe model assumption and MLEs of the parameters, whereas Exact distribution of MLEs are derived in Section 3. Different types of confidence intervals *viz.* exact, bootstrap confidence intervals are constructed in Section 4. Further we carry out the Bayesian analysis of the data in Section 5. An extensive simulation study is reported along with a particular data set for data analysis in Section 6. We conclude the chapter in Section 7. All the proofs of the results are given in Appendix 7.

2 Model assumption & MLEs

Before we proceed for model description let us define the integers $n, m(< n)$ and a set of non negative integers $(R_1, \ldots, R_m)$ such that $m + \sum_{i=1}^{m} R_i = n$. Consider an experiment starts with n units under first stress level. The environment of the experiment changes to a second stressed level at a fixed time point τ. For $i \in \{1, 2, \ldots, m\}$, R_i units are removed from the remaining $(n - i - R_1 - \ldots - R_{i-1})$ units at the failure time of i-th unit of the experiment. The experiment stops running as soon as the m-th unit fails. At each stress level there are two causes of failures. For $i, j = 1, 2$, let X_{ij} denote the random variable associated with i-th risk in j-th stress level of the experiment. Further it is assumed that the causes of failures are following Cox's latent failure time model and stress level for each competing risk factor changes under the assumption of CEM. One would observe the failure time points and the cause associated with each failure. At first stress level failure time points will be observed in the form of random variable $Z_1 = \min\{X_{11}, X_{21}\}$ and $Z_2 = \min\{X_{12}, X_{22}\}$ at the second stress level. We define a random variable Z as,

$$Z = \begin{cases} Z_1, & \text{at first stress level} \\ Z_2, & \text{at second stress level} \end{cases}$$

Thus Z is the random variable denoting the failure time of the units. Let us denote by $\mathcal{D}_1$ and $\mathcal{D}_2$ the observed data sets in first and second stress level, respectively. Thus if D is the random variable denoting the number of failures in first stress level then $\mathcal{D}_1$ and $\mathcal{D}_2$ are of the following form,

$\mathcal{D}_1 = \{(z_{1:m:n}, R_1, \delta_{1,1}), (z_{2:m:n}, R_2, \delta_{2,1}), \ldots, (z_{d:m:n}, R_d, \delta_{d,1})\}$

$\mathcal{D}_2 = \{(z_{d+1:m:n}, R_{d+1}, \delta_{d+1,2}), (z_{d+2:m:n}, R_{d+2}, \delta_{d+2,2}), \ldots, (z_{m:m:n}, R_m, \delta_{m,2})\}$

where,

$d:$ is the number of failures observed in the first stress level.

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i \text{ -th failure comes from first cause in } j \text{ -th stress level,} \\ 0, & \text{if } i \text{ -th failure comes from second cause in } j \text{ -th stress level.} \end{cases}$$

Clearly the complete data of the experiment is $\{\mathcal{D}_1, \mathcal{D}_2\}$.

It is to be noted that, for $z_{i:m:n} < \tau$, the likelihood contribution at the observation $(z_{i:m:n}, \delta_{i,1} = 1)$ is obtained as,

$$\begin{aligned} L(\theta_{11}, \theta_{21}, \theta_{12}, \theta_{22} | (z_{i:m:n}, \delta_{i,1} = 1)) =& \frac{1}{\theta_{11}} e^{-\frac{1}{\theta_{11}} z_{i:m:n}(1+R_i)} \; e^{-\frac{1}{\theta_{21}} z_{i:m:n}(1+R_i)} \\ =& \frac{1}{\theta_{11}} e^{-(\frac{1}{\theta_{11}} + \frac{1}{\theta_{21}}) z_{i:m:n}(1+R_i)}. \end{aligned} \tag{2}$$

Similarly, the likelihood contribution at the observation $(z_{i:m:n}, \delta_{i,1} = 0)$ is obtained as,

$$L(\theta_{11}, \theta_{21}, \theta_{12}, \theta_{22} | (z_{i:m:n}, \delta_{i,1} = 0)) = \frac{1}{\theta_{21}} e^{-(\frac{1}{\theta_{11}} + \frac{1}{\theta_{21}}) z_{i:m:n}(1+R_i)}. \tag{3}$$

On the other hand, for $\tau < z_{i:m:n}$, the likelihood contribution at the observation $(z_{i:m:n}, \delta_{i,2} = 1)$ is obtained as,

$$\begin{aligned} L(\theta_{11}, \theta_{21}, \theta_{12}, \theta_{22} | (z_{i:m:n}, \delta_{i,2} = 1)) =& \frac{1}{\theta_{12}} e^{-\frac{1}{\theta_{12}}(z_{i:m:n} - \tau) - \frac{\tau}{\theta_{11}}} \; e^{-\frac{1}{\theta_{22}}(z_{i:m:n} - \tau) - \frac{\tau}{\theta_{21}}} \\ =& \frac{1}{\theta_{12}} e^{-(\frac{1}{\theta_{12}} + \frac{1}{\theta_{22}})(z_{i:m:n} - \tau) - \tau(\frac{1}{\theta_{11}} + \frac{1}{\theta_{21}})}. \end{aligned} \tag{4}$$

Similarly, for $\tau < z_{i:m:n}$, the likelihood contribution at the observation $(z_{i:m:n}, \delta_{i,2} = 0)$ is obtained as,

$$L(\theta_{11}, \theta_{21}, \theta_{12}, \theta_{22} | (z_{i:m:n}, \delta_{i,2} = 1)) = \frac{1}{\theta_{22}} e^{-(\frac{1}{\theta_{12}} + \frac{1}{\theta_{22}})(z_{i:m:n} - \tau) - \tau(\frac{1}{\theta_{11}} + \frac{1}{\theta_{21}})}. \tag{5}$$

Based on the data, let us denote by,

$$w_1 = \sum_{i=1}^{d} z_{i:m:n}\big(1 + R_i\big) + \tau\Big[\big(m - d\big) + \sum_{i=d+1}^{m} R_i\Big] \text{ and } w_2 = \sum_{i=d+1}^{m} (z_{i:m:n} - \tau)(1 + R_i).$$

For $i, j = 1, 2$, suppose D_{ij} is the random variable denoting the number of failures occurring due to cause i in the j -th stress level. The likelihood function from the expressions (2) to (5) and based on the data, can be written as,

$$L(\theta_{11}, \theta_{21}, \theta_{12}, \theta_{22} | data) = c\Big(\frac{1}{\theta_{11}}\Big)^{d_{11}} \Big(\frac{1}{\theta_{21}}\Big)^{d - d_{11}} \Big(\frac{1}{\theta_{12}}\Big)^{d_{12}} \Big(\frac{1}{\theta_{22}}\Big)^{m - d - d_{12}} e^{-\frac{w_1}{\theta_{.1}} - \frac{w_2}{\theta_{.2}}}, \tag{6}$$

where, c is the normalizing constant, independent of the parameters and for $s = 1, 2$, $\dfrac{1}{\theta_{.s}} = \dfrac{1}{\theta_{1s}} + \dfrac{1}{\theta_{2s}}$. Derivation of c is supplied in Appendix 7. For $i, j = 1, 2$, MLEs of θ_{ij} are obtained by maximizing the likelihood (or equivalently the log likelihood) function. Note that MLE of θ_{ij} exists only if $D_{ij} > 0$ and it is given by,

$$\widehat{\theta}_{ij} = \frac{W_j}{D_{ij}}, \quad i, j = 1, 2. \tag{7}$$

3 Conditional distribution of MLEs

In this section, we derive conditional distribution of the MLEs of the parameters. Let us define an event $\mathcal{D}^* = \{D_{11} > 0, D_{21} > 0, D_{12} > 0, D_{22} > 0\}$. Note that MLEs of all the parameters exist only if the event $\mathcal{D}^*$ occurs. For $s = 1, 2$, the distribution functions of $\widehat{\theta}_{s1}$ and $\widehat{\theta}_{s2}$ are given below.

Theorem 1. *The distribution function of* $\widehat{\theta}_{s1}$ *is given by,*

$$\begin{aligned}
F_{\widehat{\theta}_{s1|\mathcal{D}^*}}(x) =& P(\widehat{\theta}_{s1} \leq x | D^*) \\
=& \frac{c}{P(\mathcal{D}^*)} \sum_{d=2}^{m-2} \sum_{i=1}^{d-1} \Bigg[\binom{d}{i} \Big(\frac{\theta_{s1}}{\theta_{11} + \theta_{21}} \Big)^{d-i} \Big(1 - \frac{\theta_{s1}}{\theta_{11} + \theta_{21}} \Big)^{i} \Big[1 - \Big(\frac{\theta_{22}}{\theta_{12} + \theta_{22}} \Big)^{m-d} - \\
& \Big(\frac{\theta_{12}}{\theta_{12} + \theta_{22}} \Big)^{m-d} \Big] \frac{1}{\prod_{j=1}^{m-d} \sum_{p=j}^{m-d} (1 + R_p)} \times \\
& \sum_{l=0}^{d} \Big[\frac{(-1)^l e^{-\frac{\tau}{\theta_{.1}} \left[l + m - d + \sum_{j=d-l+1}^{m} R_j \right]}}{\left[\prod_{j=1}^{l} \sum_{p=d-l+1}^{d-l+j} (1 + R_p) \right] \left[\prod_{j=1}^{d-l} \sum_{p=j}^{d-l} (1 + R_p) \right]} \times \\
& F_G \Big(x; \frac{\tau}{i} \big[l + m - d + \sum_{j=d-l+1}^{m} R_j \big], d, \frac{i}{\theta_{.1}} \Big) \Big] \Bigg].
\end{aligned} \tag{8}$$

Proof. See Appendix 7. ∎

Corollary 1.1. *The probability density function of* $\widehat{\theta}_{s1}$ *is given by,*

$$f_{\widehat{\theta}_{s1|\mathcal{D}^*}}(x) = \frac{c}{P(\mathcal{D}^*)} \sum_{d=2}^{m-2} \sum_{i=1}^{d-1} \Bigg[\binom{d}{i} \Big(\frac{\theta_{s1}}{\theta_{11} + \theta_{21}} \Big)^{d-i} \Big(1 - \frac{\theta_{s1}}{\theta_{11} + \theta_{21}} \Big)^{i} \Big[1 - \Big(\frac{\theta_{22}}{\theta_{12} + \theta_{22}} \Big)^{m-d} -$$

$$\Big(\frac{\theta_{12}}{\theta_{12}+\theta_{22}}\Big)^{m-d}\Big]\frac{1}{\prod_{j=1}^{m-d}\sum_{p=j}^{m-d}(1+R_p)}\times$$
$$\sum_{l=0}^{d}\Big[\frac{(-1)^l e^{-\frac{\tau}{\theta_{.1}}\big[l+m-d+\sum_{j=d-l+1}^{m}R_j\big]}}{\big[\prod_{j=1}^{l}\sum_{p=d-l+1}^{d-l+j}(1+R_p)\big]\big[\prod_{j=1}^{d-l}\sum_{p=j}^{d-l}(1+R_p)\big]}\times$$
$$f_G\Big(x;\frac{\tau}{i}\big[l+m-d+\sum_{j=d-l+1}^{m}R_j\big],d,\frac{i}{\theta_{.1}}\Big)\Big]\Big]. \quad (9)$$

Theorem 2. *For $s=1,2$, the distribution function of $\widehat{\theta}_{s2}$ is given by,*

$$\begin{aligned}F_{\widehat{\theta}_{s2}|D^*}(x)=&P(\widehat{\theta}_{s2}\le x|D^*)\\
=&\frac{c}{P(\mathcal{D}^*)}\sum_{d=2}^{m-2}\sum_{k=1}^{m-d-1}\Bigg[\Big[1-\Big(\frac{\theta_{21}}{\theta_{11}+\theta_{21}}\Big)^d-\Big(\frac{\theta_{11}}{\theta_{11}+\theta_{21}}\Big)^d\Big]\binom{m-d}{k}\times\\
&\Big(1-\frac{\theta_{s2}}{\theta_{12}+\theta_{22}}\Big)^k\Big(\frac{\theta_{s2}}{\theta_{12}+\theta_{22}}\Big)^{m-d-k}\frac{1}{\prod_{j=1}^{m-d}\sum_{p=j}^{m-d}(1+R_p)}\times\\
&\sum_{l=0}^{d}\frac{(-1)^l e^{-\frac{\tau}{\theta_{.1}}\big[l+m-d+\sum_{i=d-l+1}^{m}R_i\big]}}{\big[\prod_{j=1}^{l}\sum_{p=d-l+1}^{d-l+j}(1+R_p)\big]\big[\prod_{j=1}^{d-l}\sum_{p=j}^{d-l}(1+R_p)\big]}F_G\Big(x;0,m-d,\frac{k}{\theta_{.2}}\Big)\Bigg].\end{aligned} \quad (10)$$

Proof. See Appendix 7. ∎

Corollary 2.1. *For $s=1,2$, the probability density function of $\widehat{\theta}_{s2}$ is given by,*

$$\begin{aligned}f_{\widehat{\theta}_{s2}|D^*}(x)=&\frac{c}{P(\mathcal{D}^*)}\sum_{d=2}^{m-2}\sum_{k=1}^{m-d-1}\Bigg[\Big[1-\Big(\frac{\theta_{21}}{\theta_{11}+\theta_{21}}\Big)^d-\Big(\frac{\theta_{11}}{\theta_{11}+\theta_{21}}\Big)^d\Big]\binom{m-d}{k}\times\\
&\Big(1-\frac{\theta_{s2}}{\theta_{12}+\theta_{22}}\Big)^k\Big(\frac{\theta_{s2}}{\theta_{12}+\theta_{22}}\Big)^{m-d-k}\frac{1}{\prod_{j=1}^{m-d}\sum_{p=j}^{m-d}(1+R_p)}\times\\
&\sum_{l=0}^{d}\frac{(-1)^l e^{-\frac{\tau}{\theta_{.1}}\big[l+m-d+\sum_{i=d-l+1}^{m}R_i\big]}}{\big[\prod_{j=1}^{l}\sum_{p=d-l+1}^{d-l+j}(1+R_p)\big]\big[\prod_{j=1}^{d-l}\sum_{p=j}^{d-l}(1+R_p)\big]}f_G\Big(x;0,m-d,\frac{k}{\theta_{.2}}\Big)\Bigg].\end{aligned} \quad (11)$$

4 Confidence intervals

In this section we construct exact and bootstrap confidence intervals (CI) of the parameters of interest. They are discussed below.

4.1 Exact confidence interval

For any $\alpha \in (0,1)$, if θ_{ij}^L and θ_{ij}^U denote $100(1-\alpha)\%$ lower and upper confidence limits of θ_{ij} then they are obtained by solving for θ_{ij} the following two equations:

$$F_{\widehat{\theta}_{ij}|D^*}(\widehat{\theta}_{ij}^{obs}) = 1 - \frac{\alpha}{2}, \tag{12}$$

$$F_{\widehat{\theta}_{ij}|D^*}(\widehat{\theta}_{ij}^{obs}) = \frac{\alpha}{2}. \tag{13}$$

Here $\widehat{\theta}_{ij}^{obs}$ denotes the observed MLE of θ_{ij}. Note that Equations (12) and (13) involve other parameters $\theta_{i^{'}j^{'}}$; $(i^{'}j^{'}) \neq (ij)$, all of them are replaced by their observed values of MLEs respectively. It is to be noted solutions of the above equations exist if the following two properties hold true:

Property-1:

For any $x > 0$ and $i, j = 1, 2$, the function $F_{\widehat{\theta}_{ij}|D^*}(x)$ is a monotonically decreasing function of θ_{ij}.

Property-2:

For any $x > 0$, $\lim_{\theta_{ij}\to 0} F_{\widehat{\theta}_{ij}|D^*}(x) = 1$ and $\lim_{\theta_{ij}\to\infty} F_{\widehat{\theta}_{ij}|D^*}(x) = 0$.

Since we could not establish Property-1 and Property-2 analytically, a graphical display of the functions (as shown in for data analysis) reveals that these two properties hold true and hence can be used to construct exact confidence interval of the parameters. Equations (12) and (13) are non linear equations and one needs to solve them by non linear solvers *viz.* bisection, Newton Rapshon method etc.

4.2 Bootstrap confidence interval

The other confidence interval *viz.* percentile bootstrap confidence intervals can be constructed using the following algorithm.

Algorithm 1:

Step-1: Given the values of $n, m, R_1, R_2, \ldots, R_m$ and T, generate sample and estimate $\widehat{\theta}_{11}, \widehat{\theta}_{21}, \widehat{\theta}_{12}, \widehat{\theta}_{22}$ from equation (7).

Step-2: Using the same values of $n, m, R_1, R_2, \ldots, R_m, T$, generate sample with $\widehat{\theta}_{11}, \widehat{\theta}_{21}, \widehat{\theta}_{12}, \widehat{\theta}_{22}$ obtained in Step-1 and estimate values of the parameters say,$\widehat{\theta}^*_{11}, \widehat{\theta}^*_{21}, \widehat{\theta}^*_{12}, \widehat{\theta}^*_{22}$ from equation (7).

Step-3: Repeat Step-2 a large number of times say, M times and obtain $\widehat{\theta}_{ij}$, M times. Arrange them in an increasing order to obtain $\widehat{\theta}^{*(1)}_{ij} < \widehat{\theta}^{*(2)}_{ij} < \ldots < \widehat{\theta}^{*(M)}_{ij}$.

Step-4: For $\alpha \in (0,1)$, a $100(1-\alpha)\%$ percentile bootstrap confidence interval for θ_{ij} is then given by, $\left(\widehat{\theta}^{*([M\frac{\alpha}{2}])}_{ij}, \widehat{\theta}^{*([M(1-\frac{\alpha}{2})]}_{ij}\right)$, where $[x]$ denotes the largest integer less than or equal to x.

5 Bayesian Analysis

In this section we carry out Bayesian analysis of the data coming from the model under consideration. We obtain Bayes estimates of the parameters and the associated credible intervals under square error loss functions. Before we proceed further, let us denote Beta-Gamma distribution with parameters $b_0 > 0$, $a_0 > 0$, $a_1 > 0$, $a_2 > 0$ by $BG(b_0, a_0, a_1, a_2)$ with the following density function.

$$f(x,y) = b_0^{a_0} \frac{\Gamma(a_1+a_2)}{\Gamma(a_0)\Gamma(a_1)\Gamma(a_2)} e^{b_0(x+y)} x^{a_1-1} y^{a_2-1} (x+y)^{a_0-a_1-a_2}, \quad x>0, y>0. \tag{14}$$

For further details on Beta-Gamma distribution, one may see Penna and Gupta [40]. We make the re-parametrization of the parameters as $\lambda_{ij} = \frac{1}{\theta_{ij}}$, for $i,j = 1,2$. Further, we assume that the parameters of the model have the following prior distributions.

For $b_{01} > 0$, $a_{01} > 0$, $a_{11} > 0$, $a_{21} > 0$, $b_{02} > 0$, $a_{02} > 0$, $a_{12} > 0$, $a_{22} > 0$,

$(\lambda_{11}, \lambda_{21}) \sim BG(b_{01}, a_{01}, a_{11}, a_{21})$ and $(\lambda_{12}, \lambda_{22}) \sim BG(b_{02}, a_{02}, a_{12}, a_{22})$ and $(\lambda_{11}, \lambda_{21})$ are independent of $(\lambda_{12}, \lambda_{22})$.

5.1 Posterior distribution

The posterior distribution of the re-parametrized parameters $(\lambda_{11}, \lambda_{21}, \lambda_{12}, \lambda_{22})$ turn out to be in the following form.

$$\pi(\lambda_{11}, \lambda_{21}, \lambda_{12}, \lambda_{22}|data) \propto \lambda_{11}^{d_{11}+a_{11}-1}\lambda_{21}^{d-d_{11}+a_{21}-1}e^{-(b_{01}+w_1)(\lambda_{11}+\lambda_{21})}\lambda_{12}^{d_{12}+a_{12}-1}\lambda_{22}^{m-d-d_{12}+a_{22}-1}$$
$$e^{-(b_{02}+w_2)(\lambda_{12}+\lambda_{22})}(\lambda_{11}+\lambda_{21})^{a_{01}-a_{11}-a_{21}}(\lambda_{12}+\lambda_{22})^{a_{02}-a_{12}-a_{22}}, \lambda_{11} > 0, \lambda_{21} > 0, \lambda_{12} > 0, \lambda_{22} > 0 \quad (15)$$

Under square error loss function, the Bayes estimators of θ_{ij} are obtained as, $\widehat{\theta}_{ij} =$ for $i, j = 1, 2$.

5.2 Credible interval (CRI)

We now consider construction of symmetric and highest posterior density (HPD) credible intervals of the parameters. Note that, from 5.1, the joint posterior distribution of $\lambda_{11}, \lambda_{21}$ is obtained as, $(\lambda_{11}, \lambda_{21}|data) \sim BG(b_{01} + w_1, a_{01} + d, a_{11} + d_{11}, a_{21} + d - d_{11})$ and the joint posterior distribution of $\lambda_{12}, \lambda_{22}$ is obtained as, $(\lambda_{12}, \lambda_{22}|data) \sim BG(b_{02} + w_2, a_{02} + m - d, a_{12} + d_{12}, a_{22} + m - d - d_{12})$. We need the following lemma, to proceed further.

lemma 1. *Two random variables X and Y follow $BG(b_0, a_0, a_1, a_2)$ if and only if, $\frac{X}{X+Y} \sim Beta(a_1, a_2)$ and $X + Y \sim Gamma(a_0, b_0)$.*

Proof. The proof is straight forward and hence is not provided. ∎

We now provide an algorithm to construct $100(1 - \alpha)\%$ symmetric and HPD credible intervals of θ_{11} and θ_{21}. Similar methods can be used for θ_{12} and θ_{22} also.

Algorithm 2:

Step-1: Generate $\lambda_{11}+\lambda_{21}$ from $Gamma(a_{01}+d, b_{01}+w_1)$. Generate $\frac{\lambda_{11}}{\lambda_{11}+\lambda_{21}}$ from $Beta(a_{11}+d_{11}, a_{21}+d-d_{11})$.

Step-2: Obtain λ_{11} and λ_{21} from Step-1. Obtain $\theta_{11}=\frac{1}{\lambda_{11}}$ and $\theta_{21}=\frac{1}{\lambda_{21}}$.

Step-3: Repeat Step-1 and Step-2 quite a large number of times say, M, to obtain M number of θ_{11} and θ_{21}. Arrange M values of θ_{11} and θ_{21} each in increasing order as, $\theta_{11}^1 \leq \theta_{11}^2 \ldots \leq \theta_{11}^M$ and $\theta_{21}^1 \leq \theta_{21}^2 \ldots \leq \theta_{21}^M$ respectively.

Step-4: The $100(1-\alpha)\%$ symmetric and HPD credible intervals of θ_{11} are obtained as $(\theta_{11}^{[M\frac{\alpha}{2}]}, \theta_{11}^{[M(1-\frac{\alpha}{2})]})$ and $(\theta_{11}^{j^*}, \theta_{11}^{j^*+[M(1-\alpha)]})$, respectively, where, $j^* \in \{1,2,\ldots,[M\alpha]\}$ is an integer such that, $\theta_{11}^{j^*+[M(1-\alpha)]} - \theta_{11}^{j^*} \leq \theta_{11}^{j+[M(1-\alpha)]} - \theta_{11}^{j}$ for $\forall j = 1,2,\ldots,[M\alpha]$ and $[x]$ is the largest integer not exceeding x.

6 Simulation results

In this section we carry out simulation study of our model. We compute exact and bootstrap confidence intervals as well as symmetric and HPD credible intervals of the parameters of the interest. They are computed at 5% and 1% levels of significance. We have taken $\theta_{11}=1.3, \theta_{21}=1.1, \theta_{12}=0.7, \theta_{22}=0.5$ as our designing parameters. For Bayesian analysis the hyper parameters are taken as $b_{01}=b_{02}=0,\ a_{01}=a_{02}=2,\ a_{11}=a_{21}=a_{12}=a_{22}=1$. These values are chosen such that, they match with the corresponding MLEs of the parameters. Thus a comparison study can be made between the performance of frequentest and Bayesian analysis of data in our model. Different values of n and τ are considered to check for repeating performances of the proposed model under different censoring schemes. For our study we have considered three different censoring schemes and they are described as:- Scheme-1: $R_1 = n - m, R_2 = \ldots = R_m = 0$, Scheme-2: $R_1 = \ldots = R_{m/2-1} = 0, R_{m/2} = n - m, R_{m/2+1} = \ldots = R_m = 0$, Scheme-3: $R_1 = (n-m)/2, R_2 = \ldots = R_{m-1} = 0, R_m = (n-m)/2$. In all these cases, $m + \sum_{i=1}^{m} R_i = n$.

Table 1: Classical results for θ_{11} with $\theta_{11} = 1.3, \theta_{21} = 1.1, \theta_{12} = 0.7, \theta_{22} = 0.5$

n	m	τ	Scheme	Bias	MSE	Exact CI		Bootstrap CI	
						95%	99%	95%	99%
35	30	0.5	I	0.133	0.332	2.839 (96.02)	4.255 (99.40)	3.554 (94.20)	5.851 (98.32)
35	30	0.6	I	0.126	0.297	2.534 (96.37)	3.879 (99.35)	3.149 (94.75)	5.324 (98.62)
35	30	0.7	I	0.125	0.283	2.349 (95.57)	3.504 (99.57)	2.982 (93.77)	4.895 (98.12)
40	34	0.5	I	0.130	0.314	2.563 (95.92)	3.786 (99.47)	3.151 (93.95)	5.229 (98.40)
40	34	0.6	I	0.123	0.278	2.286 (95.42)	3.380 (99.45)	2.806 (94.05)	4.703 (98.30)
40	34	0.7	I	0.106	0.252	2.099 (94.82)	3.076 (99.25)	2.513 (94.42)	4.174 (98.50)
35	30	0.5	II	0.134	0.321	2.626 (95.90)	3.884 (99.42)	3.270 (94.17)	5.463 (98.27)
35	30	0.6	II	0.119	0.291	2.381 (95.50)	3.584 (99.40)	2.896 (94.62)	4.854 (98.55)
35	30	0.7	II	0.114	0.267	2.228 (95.20)	3.334 (99.35)	2.669 (94.75)	4.481 (98.70)
40	34	0.5	II	0.111	0.267	2.280 (95.42)	3.400 (99.37)	2.943 (93.40)	4.902 (98.27)
40	34	0.6	II	0.107	0.249	2.122 (95.02)	3.078 (99.47)	2.559 (94.62)	4.246 (98.45)
40	34	0.7	II	0.101	0.255	2.048 (94.92)	2.992 (99.02)	2.310 (94.80)	3.747 (98.35)
35	30	0.5	III	0.135	0.319	2.614 (95.67)	3.971 (99.40)	3.185 (95.05)	5.379 (98.67)
35	30	0.6	III	0.119	0.273	2.313 (95.45)	3.507 (99.40)	2.840 (94.35)	4.801 (98.67)
35	30	0.7	III	0.111	0.274	2.193 (95.07)	3.269 (99.27)	2.637 (94.30)	4.423 (98.67)
40	34	0.5	III	0.107	0.273	2.296 (95.62)	3.403 (99.57)	2.695 (94.25)	4.518 (98.70)
40	34	0.6	III	0.107	0.245	2.104 (95.70)	3.108 (99.45)	2.449 (94.15)	4.035 (98.45)
40	34	0.7	III	0.098	0.244	1.979 (94.80)	2.879 (99.30)	2.265 (94.25)	3.699 (98.90)

Table 2: Classical results for θ_{21} with $\theta_{11} = 1.3, \theta_{21} = 1.1, \theta_{12} = 0.7, \theta_{22} = 0.5$

n	m	τ	Scheme	Bias	MSE	Exact CI		Bootstrap CI	
						95%	99%	95%	99%
35	30	0.5	I	0.106	0.222	2.056 (94.82)	3.092 (99.47)	2.562 (93.55)	4.280 (98.37)
35	30	0.6	I	0.097	0.197	1.857 (94.82)	2.786 (99.47)	2.206 (94.12)	3.619 (98.95)
35	30	0.7	I	0.093	0.180	1.729 (94.57)	2.523 (99.05)	1.921 (95.25)	3.288 (98.67)
40	34	0.5	I	0.089	0.183	1.814 (94.75)	2.692 (99.25)	2.149 (94.55)	3.638 (98.37)
40	34	0.6	I	0.083	0.174	1.663 (94.92)	2.413 (99.35)	1.906 (94.65)	2.975 (98.92)
40	34	0.7	I	0.075	0.150	1.547 (95.07)	2.244 (99.10)	1.742 (94.70)	2.690 (98.90)
35	30	0.5	II	0.086	0.196	1.864 (94.65)	2.831 (99.32)	2.178 (94.70)	3.692 (98.45)
35	30	0.6	II	0.082	0.171	1.727 (95.17)	2.559 (99.32)	1.941 (94.27)	3.152 (98.82)
35	30	0.7	II	0.079	0.164	1.646 (94.70)	2.391 (99.37)	1.803 (94.90)	2.887 (99.20)
40	34	0.5	II	0.076	0.165	1.670 (95.12)	2.462 (99.30)	1.895 (94.27)	3.113 (98.67)
40	34	0.6	II	0.069	0.141	1.543 (95.15)	2.270 (99.10)	1.709 (94.60)	2.686 (98.77)
40	34	0.7	II	0.067	0.134	1.473 (94.20)	2.145 (98.72)	1.607 (95.22)	2.443 (98.82)
35	30	0.5	III	0.092	0.195	1.873 (94.82)	2.790 (99.37)	2.285 (94.40)	3.730 (98.85)
35	30	0.6	III	0.087	0.173	1.722 (94.95)	2.508 (99.22)	2.002 (94.52)	3.318 (98.70)
35	30	0.7	III	0.083	0.145	1.609 (95.37)	2.348 (99.20)	1.815 (95.50)	2.920 (98.50)
40	34	0.5	III	0.086	0.179	1.738 (95.67)	2.523 (99.47)	1.961 (95.02)	3.232 (98.70)
40	34	0.6	III	0.081	0.154	1.577 (94.25)	2.249 (98.82)	1.780 (94.55)	2.794 (98.77)
40	34	0.7	III	0.078	0.130	1.490 (95.10)	2.124 (98.97)	1.599 (95.52)	2.507 (98.75)

Table 3: Classical results for θ_{12} with $\theta_{11} = 1.3, \theta_{21} = 1.1, \theta_{12} = 0.7, \theta_{22} = 0.5$

n	m	τ	Scheme	Bias	MSE	Exact CI		Bootstrap CI	
						95%	99%	95%	99%
35	30	0.5	I	0.030	0.094	2.318 (97.70)	3.854 (99.77)	2.078 (94.95)	3.263 (99.10)
35	30	0.6	I	0.031	0.092	2.755 (97.80)	4.352 (99.55)	2.187 (95.35)	3.352 (99.25)
35	30	0.7	I	0.040	0.079	3.022 (98.25)	4.833 (99.65)	2.232 (95.17)	3.339 (99.15)
40	34	0.5	I	0.024	0.094	2.107 (97.42)	3.461 (99.77)	1.938 (94.95)	3.118 (99.02)
40	34	0.6	I	0.027	0.098	2.503 (97.40)	4.069 (99.67)	2.090 (95.82)	3.272 (99.32)
40	34	0.7	I	0.033	0.094	2.943 (97.47)	4.627 (99.75)	2.212 (95.75)	3.376 (99.27)
35	30	0.5	II	0.039	0.096	2.709 (97.35)	4.265 (99.70)	2.178 (95.05)	3.353 (98.92)
35	30	0.6	II	0.046	0.090	3.041 (97.62)	4.794 (99.65)	2.278 (95.97)	3.411 (99.25)
35	30	0.7	II	0.050	0.084	3.441 (97.52)	5.115 (99.67)	2.248 (95.37)	3.314 (98.87)
40	34	0.5	II	0.031	0.102	2.384 (96.55)	4.003 (99.75)	2.117 (95.07)	3.309 (98.82)
40	34	0.6	II	0.039	0.094	2.774 (97.12)	4.571 (99.70)	2.187 (95.95)	3.334 (99.32)
40	34	0.7	II	0.047	0.085	3.256 (97.22)	4.973 (99.57)	2.235 (95.42)	3.349 (99.07)
35	30	0.5	III	0.041	0.092	2.742 (97.75)	4.282 (99.65)	2.163 (95.10)	3.338 (99.32)
35	30	0.6	III	0.046	0.081	3.177 (98.37)	4.852 (99.72)	2.252 (95.70)	3.387 (99.30)
35	30	0.7	III	0.053	0.075	3.658 (99.22)	5.353 (99.72)	2.204 (95.37)	3.253 (98.92)
40	34	0.5	III	0.040	0.096	2.379 (97.62)	3.852 (99.80)	2.089 (95.37)	3.294 (99.02)
40	34	0.6	III	0.041	0.099	2.977 (97.55)	4.557 (99.75)	2.185 (95.37)	3.334 (99.27)
40	34	0.7	III	0.049	0.084	3.260 (97.77)	5.141 (99.72)	2.232 (95.67)	3.332 (98.97)

Table 4: Classical results for θ_{22} with $\theta_{11} = 1.3, \theta_{21} = 1.1, \theta_{12} = 0.7, \theta_{22} = 0.5$

n	m	τ	Scheme	Bias	MSE	Exact CI		Bootstrap CI	
						95%	99%	95%	99%
35	30	0.5	I	0.027	0.044	1.204 (96.32)	2.117 (99.50)	1.103 (94.67)	1.798 (98.82)
35	30	0.6	I	0.034	0.052	1.558 (97.00)	2.702 (99.67)	1.268 (95.35)	2.038 (98.90)
35	30	0.7	I	0.037	0.051	1.863 (97.22)	3.129 (99.72)	1.387 (96.10)	2.207 (99.20)
40	34	0.5	I	0.024	0.039	1.038 (95.90)	1.761 (99.52)	0.999 (94.72)	1.615 (98.77)
40	34	0.6	I	0.029	0.046	1.319 (96.67)	2.263 (99.65)	1.157 (94.47)	1.875 (98.85)
40	34	0.7	I	0.031	0.051	1.552 (96.07)	2.764 (99.72)	1.296 (95.47)	2.074 (99.02)
35	30	0.5	II	0.030	0.053	1.427 (96.62)	2.443 (99.62)	1.207 (95.42)	1.946 (98.85)
35	30	0.6	II	0.032	0.052	1.725 (96.67)	2.911 (99.62)	1.339 (95.15)	2.132 (99.12)
35	30	0.7	II	0.039	0.054	2.082 (97.12)	3.519 (99.75)	1.484 (95.72)	2.329 (99.02)
40	34	0.5	II	0.025	0.045	1.226 (96.45)	2.062 (99.30)	1.089 (95.10)	1.774 (98.60)
40	34	0.6	II	0.028	0.049	1.523 (96.67)	2.706 (99.70)	1.258 (95.47)	2.033 (98.92)
40	34	0.7	II	0.031	0.056	1.947 (96.70)	3.294 (99.62)	1.384 (95.70)	2.197 (99.22)
35	30	0.5	III	0.027	0.048	1.525 (96.22)	2.619 (99.75)	1.219 (95.22)	1.975 (98.92)
35	30	0.6	III	0.030	0.052	1.928 (96.77)	3.390 (99.60)	1.392 (95.90)	2.215 (99.35)
35	30	0.7	III	0.033	0.056	2.369 (97.02)	3.943 (99.87)	1.536 (96.47)	2.391 (99.32)
40	34	0.5	III	0.025	0.047	1.255 (96.50)	2.135 (99.52)	1.098 (94.95)	1.787 (98.90)
40	34	0.6	III	0.030	0.053	1.603 (96.45)	2.937 (99.62)	1.277 (95.55)	2.060 (98.87)
40	34	0.7	III	0.032	0.051	2.053 (97.17)	3.449 (99.75)	1.440 (95.42)	2.274 (99.07)

Table 5: Bayesian results for θ_{11} with $\theta_{11} = 1.3, \theta_{21} = 1.1, \theta_{12} = 0.7, \theta_{22} = 0.5$

n	m	τ	Scheme	Symmetric CRI		HPD CRI	
				95%	99%	95%	99%
35	30	0.5	I	2.460 (94.70)	3.872 (99.12)	2.157 (93.35)	3.391 (98.60)
35	30	0.6	I	2.281 (94.40)	3.522 (98.82)	2.028 (93.52)	3.121 (98.50)
35	30	0.7	I	2.112 (94.92)	3.164 (98.80)	1.905 (93.62)	2.859 (98.62)
40	34	0.5	I	2.262 (94.57)	3.482 (98.92)	2.014 (93.67)	3.092 (98.82)
40	34	0.6	I	1.968 (94.77)	2.914 (98.90)	1.787 (93.32)	2.651 (98.47)
40	34	0.7	I	1.865 (95.45)	2.736 (98.92)	1.706 (93.92)	2.507 (98.52)
35	30	0.5	II	2.256 (94.62)	3.473 (98.92)	2.006 (93.30)	3.088 (98.55)
35	30	0.6	II	2.095 (94.75)	3.137 (98.92)	1.889 (93.67)	2.836 (98.42)
35	30	0.7	II	1.979 (94.95)	2.925 (99.02)	1.799 (94.45)	2.663 (98.67)
40	34	0.5	II	1.999 (95.22)	2.969 (99.07)	1.812 (94.02)	2.696 (98.82)
40	34	0.6	II	1.931 (94.00)	2.841 (98.67)	1.762 (93.80)	2.596 (98.20)
40	34	0.7	II	1.787 (94.82)	2.600 (98.65)	1.644 (94.05)	2.394 (98.30)
35	30	0.5	III	2.261 (94.35)	3.475 (98.62)	2.009 (92.95)	3.091 (98.10)
35	30	0.6	III	2.042 (94.80)	3.065 (99.12)	1.842 (93.17)	2.766 (98.77)
35	30	0.7	III	1.900 (95.55)	2.797 (99.22)	1.733 (94.02)	2.554 (98.82)
40	34	0.5	III	2.087 (95.02)	3.121 (98.92)	1.884 (94.35)	2.821 (98.57)
40	34	0.6	III	1.916 (94.72)	2.819 (99.00)	1.749 (93.80)	2.575 (98.77)
40	34	0.7	III	1.820 (95.15)	2.646 (98.92)	1.674 (94.60)	2.437 (98.85)

Table 6: Bayesian results for θ_{21} with $\theta_{11} = 1.3, \theta_{21} = 1.1, \theta_{12} = 0.7, \theta_{22} = 0.5$

n	m	τ	Scheme	Symmetric CRI		HPD CRI	
				95%	99%	95%	99%
35	30	0.5	I	1.800 (94.70)	2.711 (99.15)	1.620 (93.67)	2.440 (98.60)
35	30	0.6	I	1.630 (94.67)	2.399 (98.90)	1.486 (93.67)	2.191 (98.45)
35	30	0.7	I	1.508 (94.90)	2.192 (99.15)	1.387 (93.67)	2.019 (98.77)
40	34	0.5	I	1.597 (93.52)	2.357 (99.05)	1.347 (91.42)	2.156 (98.40)
40	34	0.6	I	1.510 (94.00)	2.192 (98.82)	1.391 (93.67)	2.021 (98.25)
40	34	0.7	I	1.404 (94.47)	2.013 (98.67)	1.304 (93.77)	1.872 (98.30)
35	30	0.5	II	1.627 (94.37)	2.397 (99.00)	1.481 (93.10)	2.187 (98.60)
35	30	0.6	II	1.544 (94.72)	2.252 (99.00)	1.418 (93.57)	2.070 (98.60)
35	30	0.7	II	1.472 (95.27)	2.126 (99.00)	1.361 (94.47)	1.968 (98.87)
40	34	0.5	II	1.514 (95.32)	2.199 (99.07)	1.394 (94.60)	2.026 (98.80)
40	34	0.6	II	1.402 (95.27)	2.013 (98.97)	1.302 (94.10)	1.870 (98.82)
40	34	0.7	II	1.329 (94.20)	1.893 (98.70)	1.240 (93.37)	1.769 (98.15)
35	30	0.5	III	1.643 (93.82)	2.437 (98.47)	1.492 (92.57)	2.215 (98.22)
35	30	0.6	III	1.522 (94.55)	2.219 (98.72)	1.398 (93.67)	2.041 (98.30)
35	30	0.7	III	1.424 (96.02)	2.050 (99.35)	1.318 (94.70)	1.900 (98.92)
40	34	0.5	III	1.517 (95.20)	2.205 (98.90)	1.396 (93.90)	2.030 (98.62)
40	34	0.6	III	1.419 (94.85)	2.039 (98.95)	1.316 (94.45)	1.893 (98.77)
40	34	0.7	III	1.346 (95.02)	1.916 (99.05)	1.256 (94.87)	1.790 (98.95)

Table 7: Bayesian results for θ_{12} with $\theta_{11} = 1.3, \theta_{21} = 1.1, \theta_{12} = 0.7, \theta_{22} = 0.5$

n	m	τ	Scheme	Symmetric CRI		HPD CRI	
				95%	99%	95%	99%
35	30	0.5	I	1.711 (93.52)	2.967 (98.65)	1.422 (91.10)	2.459 (97.70)
35	30	0.6	I	1.921 (93.77)	3.512 (98.62)	1.552 (91.05)	2.829 (97.92)
35	30	0.7	I	2.186 (93.45)	4.212 (98.60)	1.717 (90.65)	3.300 (97.55)
40	34	0.5	I	1.597 (93.52)	2.696 (98.42)	1.347 (91.42)	2.272 (97.47)
40	34	0.6	I	1.814 (94.00)	3.216 (98.72)	1.490 (91.45)	2.636 (97.90)
40	34	0.7	I	2.045 (93.92)	3.828 (98.70)	1.634 (91.02)	3.050 (97.70)
35	30	0.5	II	1.871 (93.20)	3.361 (98.67)	1.524 (90.67)	2.732 (97.75)
35	30	0.6	II	2.130 (93.87)	4.047 (98.57)	1.686 (90.20)	3.195 (97.42)
35	30	0.7	II	2.246 (93.42)	4.421 (98.17)	1.743 (89.65)	3.428 (96.80)
40	34	0.5	II	1.728 (94.52)	3.010 (98.82)	1.434 (92.10)	2.488 (98.20)
40	34	0.6	II	2.000 (93.12)	3.714 (98.15)	1.605 (90.57)	2.960 (97.30)
40	34	0.7	II	2.163 (93.20)	4.157 (98.25)	1.701 (89.52)	3.265 (96.90)
35	30	0.5	III	2.084 (94.82)	3.800 (98.52)	1.685 (93.42)	3.067 (97.85)
35	30	0.6	III	2.390 (94.62)	4.613 (98.62)	1.871 (91.85)	3.610 (98.18)
35	30	0.7	III	2.666 (94.27)	5.448 (98.95)	2.029 (90.77)	4.144 (98.02)
40	34	0.5	III	1.710 (94.07)	2.996 (98.22)	1.416 (91.25)	2.469 (97.65)
40	34	0.6	III	1.974 (93.50)	3.661 (98.32)	1.582 (90.07)	2.928 (97.40)
40	34	0.7	III	2.291 (93.55)	4.519 (98.77)	1.777 (89.45)	3.494 (98.02)

Table 8: Bayesian results for θ_{22} with $\theta_{11} = 1.3, \theta_{21} = 1.1, \theta_{12} = 0.7, \theta_{22} = 0.5$

n	m	τ	Scheme	Symmetric CRI		HPD CRI	
				95%	99%	95%	99%
35	30	0.5	I	0.877 (93.37)	1.364 (98.35)	0.774 (91.37)	1.203 (97.67)
35	30	0.6	I	0.982 (94.30)	1.590 (98.60)	0.847 (92.12)	1.370 (97.85)
35	30	0.7	I	1.149 (93.85)	1.963 (98.65)	0.961 (91.27)	1.642 (97.77)
40	34	0.5	I	0.794 (93.60)	1.202 (98.75)	0.712 (91.40)	1.078 (98.12)
40	34	0.6	I	0.912 (93.52)	1.445 (98.70)	0.798 (92.07)	1.262 (98.00)
40	34	0.7	I	1.059 (93.37)	1.755 (98.52)	0.901 (90.85)	1.492 (97.55)
35	30	0.5	II	0.966 (93.70)	1.555 (98.65)	0.837 (91.57)	1.344 (98.02)
35	30	0.6	II	1.149 (93.67)	1.973 (98.70)	0.961 (90.92)	1.646 (97.72)
35	30	0.7	II	1.282 (93.97)	2.285 (98.55)	1.051 (90.25)	1.864 (97.78)
40	34	0.5	II	0.869 (93.92)	1.346 (98.67)	0.767 (91.82)	1.190 (98.07)
40	34	0.6	II	1.010 (94.05)	1.650 (98.72)	0.867 (91.35)	1.413 (98.00)
40	34	0.7	II	1.178 (93.85)	2.026 (98.75)	0.983 (91.42)	1.688 (98.05)
35	30	0.5	III	1.083 (93.67)	1.764 (98.32)	0.931 (92.87)	1.516 (98.20)
35	30	0.6	III	1.277 (94.62)	2.216 (98.75)	1.062 (93.07)	1.837 (98.27)
35	30	0.7	III	1.558 (94.87)	2.918 (98.85)	1.244 (93.42)	2.319 (98.57)
40	34	0.5	III	0.900 (94.17)	1.418 (98.82)	0.790 (92.02)	1.242 (98.10)
40	34	0.6	III	1.040 (94.30)	1.724 (98.62)	0.886 (90.72)	1.466 (97.82)
40	34	0.7	III	1.222 (93.47)	2.156 (98.45)	1.006 (90.67)	1.768 (97.27)

The simulation results of the parameters in different schemes are reported in Table-1 to Table-8. It is observed that average bias of a parameter decreases as the effective sample size increases. For instance, as the value of τ increases, sample size in first stress level increases and hence the average bias of the estimators of the parameters θ_{11} and θ_{21} decreases for fixed n and m in a specific scheme. Also with increasing values of n and(or) m, the average bias of the parameters decreases. On the other hand if τ increases for fixed n and m in a specific scheme, the effective sample size in the second stress level decreases and hence the bias of the estimators of the parameters θ_{12} and θ_{22} increases. The average MSEs of the estimators also behave accordingly. As expected, lengths of both the exact and bootstrap confidence intervals of the parameters decrease with increasing sample sizes. The average length of exact confidence interval of the parameters in the first stress level *viz.* θ_{11} and θ_{21} are smaller than their average length of bootstrap confidence interval. However, the opposite is true for the parameters in the second stress level *viz.* θ_{12} and θ_{22}. Thus it is not evident which method of construction of confidence intervals of the parameters works better in classical analysis. However, in both the cases the corresponding coverage probabilities are matching quite well with the specified confidence coefficients. In case of Bayesian analysis, it is observed that the length of the credible intervals get improved with the increasing effective sample sizes of the experiment. The coverage percentages match quite close to the corresponding nominal values. Lengths of the credible intervals are smaller than the exact and bootstrap confidence intervals of the parameters with keeping the coverage percentages as close to the nominal percentages. It is evident that one can carry out the Bayesian analysis part with the prior distribution used in this case. Thus as a practitioner's point of view, if (s)he has the prior information to use the prior distribution given in (5), it is recommended to carry out the Bayesian analysis with the same choice of hyper parameters values mentioned before. Otherwise, choose the frequentest analysis and choose bootstrap method since it involves less computational cost than that of exact confidence interval construction method.

Data Analysis:

We generate an artificial data set by taking the set of parameters as, $\theta_{11} = 2.0, \theta_{21} = 1.5, \theta_{12} = 1.0, \theta_{22} = 0.75$. The experimental design used in generating the data is $n = 40, m = 30, \tau = 0.5, R_1 = 5, R_2 = R_3 = \ldots = R_{m-1} = 0, R_m = 5$. The data set is presented below.

Table 9: Simulated Data set

First Stress Level	Second Stress Level
8.78e-05	0.526
1.65e-04	0.582
0.031	0.588
0.067	0.625
0.124	0.704
0.127	0.721
0.155	0.733
0.164	0.761
0.174	0.805
0.197	0.851
0.212	0.881
0.224	0.883
0.231	0.935
0.242	
0.311	
0.391	
0.418	

From the data set we see that, $w_1 = 11.852, w_2 = 5.278, D_{11} = 9, D_{21} = 8, D_{12} = 6, D_{22} = 7$. Thus maximum likelihood estimates of the parameters turned out to be as, $\widehat{\theta}_{11} = 1.316$, $\widehat{\theta}_{21} = 1.481$, $\widehat{\theta}_{12} = 0.879$, $\widehat{\theta}_{22} = 0.754$. To construct exact confidence intervals of the parameters, it is assumed that $P_{\theta ij}(\widehat{\theta}_{ij} \leq x)$ is monotonically decreasing function of θ_{ij} for $i, j = 1, 2$ for any x. Here we take x to be maximum likelihood estimate of the corresponding parameter. However we could not prove this monotonic property analytically. Instead, we plot $P_{\theta ij}(\widehat{\theta}_{ij} \leq x)$ vs θ_{ij} for $i, j = 1, 2$ to check this property visually. These plots are provided in Figure-1 to Figure-4.

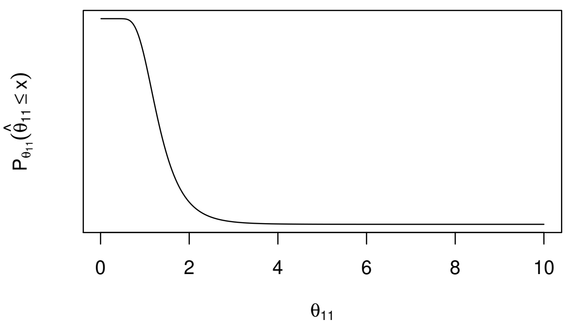

Figure 1: Stochastic monotonic property of $\widehat{\theta}_{11}$

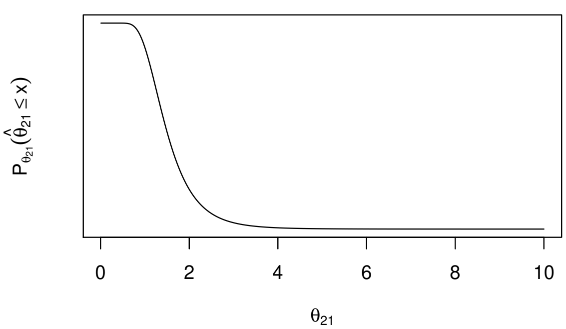

Figure 2: Stochastic monotonic property of $\widehat{\theta}_{21}$

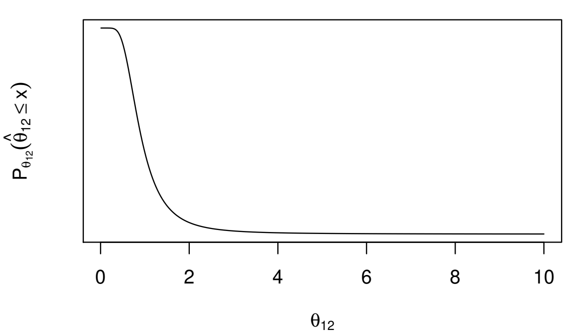

Figure 3: Stochastic monotonic property of $\widehat{\theta}_{12}$

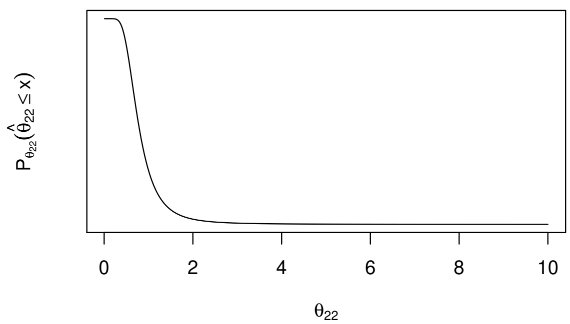

Figure 4: Stochastic monotonic property of $\widehat{\theta}_{22}$

The 95% different confidence intervals of the parameters from the simulated data set are reported below:

Table 10: Different types of confidence intervals for simulated data in Table 9

Parameter	Exact CI	Bootstrap CI	Symmetric CRI	HPD CRI
θ_{11}	(0.731, 2.636)	(0.757, 2.952)	(0.700, 2.473)	(0.620, 2.245)
θ_{21}	(0.799, 3.131)	(0.818, 3.514)	(0.754, 2.899)	(0.652, 2.617)
θ_{12}	(0.411, 2.587)	(0.370, 2.386)	(0.400, 1.918)	(0.317, 1.657)
θ_{22}	(0.371, 1.998)	(0.328, 1.934)	(0.368, 1.525)	(0.289, 1.314)

7 Conclusion

In this chapter, we have discussed progressive Type-II censoring in presence of competing risks under two stress levels. We have assumed one parameter exponential distribution for each risk factor with CEM for our analysis under the two stress levels. The MLEs of the parameters are obtained and they are conditional MLEs. We have derived exact conditional distributions of the MLEs of the parameters which was used to construct exact confidence intervals of the parameters. We also have carried out the Bayesian analysis and obtained symmetric and HPD credible intervals. In simulation study, satisfactory outputs have come in terms of bias, MSE and coverage probabilities. It is recommended to carry out Bayesian analysis to construct credible intervals of the parameters if prior information of the set of the parameters are available, otherwise use bootstrap method to construct their confidence intervals. Finally a specific simulated data set is used for illustration. It is to be commented that although the exponential distribution is assumed as parent distribution for each factor, a more general and widely applicable distribution like Weibull distribution can be used instead of that. MLEs of the parameters and derivation of their exact distributions may give serious challenges to the investigator. More work is needed along that direction.

Appendix

Derivation of constant c

The constant c in the likelihood function 6 is such that,

$$\sum_{d=0}^{m}\sum_{d_{12}=0}^{m-d}\sum_{d_{11}=0}^{d}\binom{d}{d_{11}}\binom{m-d}{d_{12}}\Big(\frac{1}{\theta_{11}}\Big)^{d_{11}}\Big(\frac{1}{\theta_{21}}\Big)^{d-d_{11}}\Big(\frac{1}{\theta_{12}}\Big)^{d_{12}}\Big(\frac{1}{\theta_{22}}\Big)^{m-d-d_{12}}\int_V e^{-\frac{w_1}{\theta_{.1}}-\frac{w_2}{\theta_{.2}}}\times \prod_{i=1}^{m} dz_{i:m:n}=\frac{1}{c},\ \text{where, } V=\{0<z_{1:m:n}<\ldots<z_{d:m:n}<\tau<z_{d+1:m:n}<\ldots<z_{m:m:n}\}. \tag{16}$$

We now calculate left hand side of above equation below.

$$\begin{aligned}
&\sum_{d=0}^{m}\sum_{d_{12}=0}^{m-d}\sum_{d_{11}=0}^{d}\binom{d}{d_{11}}\binom{m-d}{d_{12}}\Big(\frac{1}{\theta_{11}}\Big)^{d_{11}}\Big(\frac{1}{\theta_{21}}\Big)^{d-d_{11}}\Big(\frac{1}{\theta_{12}}\Big)^{d_{12}}\Big(\frac{1}{\theta_{22}}\Big)^{m-d-d_{12}}\int_V e^{-\frac{w_1}{\theta_{.1}}-\frac{w_2}{\theta_{.2}}}\times \\
&\prod_{i=1}^{m} dz_{i:m:n}\\
=&\sum_{d=0}^{m}\sum_{d_{12}=0}^{m-d}\sum_{d_{11}=0}^{d}\Bigg[\binom{d}{d_{11}}\Big(\frac{\theta_{21}}{\theta_{11}+\theta_{21}}\Big)^{d_{11}}\Big(\frac{\theta_{11}}{\theta_{11}+\theta_{21}}\Big)^{d-d_{11}}\binom{m-d}{d_{12}}\Big(\frac{\theta_{22}}{\theta_{12}+\theta_{22}}\Big)^{d_{12}}\times\\
&\Big(\frac{\theta_{12}}{\theta_{12}+\theta_{22}}\Big)^{m-d-d_{12}} e^{-\frac{\tau}{\theta_{.1}}\left[m-d+\sum\limits_{j=d+1}^{m}R_j\right]}\Big(\frac{1}{\theta_{.1}}\Big)^{d}\int_0^{\tau}\int_0^{z_{d-1:m:n}}\cdots\int_0^{z_{2:m:n}}\times\\
&e^{-\frac{1}{\theta_{.1}}\sum\limits_{j=1}^{d} z_{j:m:n}(1+R_j)}\prod_{j=1}^{d} dz_{j:m:n}\Big(\frac{1}{\theta_{.2}}\Big)^{(m-d)}\int_{\tau}^{z_{m:m:n}}\cdots\int_{\tau}^{z_{d+2:m:n}} e^{-\frac{1}{\theta_{.2}}\sum\limits_{j=d+1}^{m}(z_{j:m:n}-\tau)(1+R_j)}\times\\
&\prod_{j=1}^{d+1} dz_{j:m:n}\Bigg]\\
=&\sum_{d=0}^{m}\sum_{d_{12}=0}^{m-d}\sum_{d_{11}=0}^{d}\Bigg[\binom{d}{d_{11}}\Big(\frac{\theta_{21}}{\theta_{11}+\theta_{21}}\Big)^{d_{11}}\Big(\frac{\theta_{11}}{\theta_{11}+\theta_{21}}\Big)^{d-d_{11}}\binom{m-d}{d_{12}}\Big(\frac{\theta_{22}}{\theta_{12}+\theta_{22}}\Big)^{d_{12}}\times\\
&\Big(\frac{\theta_{12}}{\theta_{12}+\theta_{22}}\Big)^{m-d-d_{12}}\sum_{l=0}^{d}\frac{(-1)^l e^{-\frac{\tau}{\theta_{.1}}\left[l+m-d+\sum_{j=d-l+1}^{m}R_j\right]}}{\left[\prod_{j=1}^{l}\sum_{p=d-l+1}^{d-l+j}(1+R_p)\right]\left[\prod_{j=1}^{d-l}\sum_{p=j}^{d-l}(1+R_p)\right]}\times\\
&\frac{1}{\prod_{j=1}^{m-d}\sum_{p=j}^{m-d}(1+R_p)}\Bigg]
\end{aligned}$$

(by using Lemma 1 from [7])

$$=\sum_{d=0}^{m}\sum_{l=0}^{d}\frac{(-1)^l e^{-\frac{\tau}{\theta_{.1}}\left[l+m-d+\sum_{j=d-l+1}^{m}R_j\right]}}{\left[\prod_{j=1}^{l}\sum_{p=d-l+1}^{d-l+j}(1+R_p)\right]\left[\prod_{j=1}^{d-l}\sum_{p=j}^{d-l}(1+R_p)\right]}\frac{1}{\prod_{j=1}^{m-d}\sum_{p=j}^{m-d}(1+R_p)}.$$

Thus the constant c turns out to be,

$$c=\left[\sum_{d=0}^{m}\sum_{l=0}^{d}\frac{(-1)^l e^{-\frac{\tau}{\theta_{.1}}\left[l+m-d+\sum_{j=d-l+1}^{m}R_j\right]}}{\left[\prod_{j=1}^{l}\sum_{p=d-l+1}^{d-l+j}(1+R_p)\right]\left[\prod_{j=1}^{d-l}\sum_{p=j}^{d-l}(1+R_p)\right]}\frac{1}{\prod_{j=1}^{m-d}\sum_{p=j}^{m-d}(1+R_p)}\right]^{-1}.$$

Next we prove Theorem 1 and Theorem 2. To do so, we find the conditional MGFs of the MLEs of the parameters. Here we provide the derivation of MGFs of $\widehat{\theta}_{11}$ and $\widehat{\theta}_{12}$. The same calculations can be carried out to find the conditional MGFs of $\widehat{\theta}_{21}$ and $\widehat{\theta}_{22}$. In these derivations, we will use Lemma 1, given by Balakrishnan *et al* in [7]. Before we proceed for the derivations, we note the following conditional distribution,

for $d\in\{0,1,\dots,m\}, i\in\{0,1,\dots,d\}, k\in\{0,1,\dots,m-d\}$, the conditional distribution of $(Z_{1:m:n},Z_{2:m:n},\dots,Z_{d:m:n},\dots,Z_{m:m:n})$, conditioning on the event $(D=d,D_{11}=i,D_{12}=k)$ is obtained from the likelihood function 6 and is given by,

$$f_{Z_{1:m:n},\dots,Z_{m:m:n}|(D=d,D_{11}=i,D_{12}=k}(z_{1:m:n},\dots,z_{m:m:n})$$
$$=\frac{c\binom{d}{i}\binom{m-d}{j}\left(\frac{1}{\theta_{11}}\right)^i\left(\frac{1}{\theta_{21}}\right)^{d-i}\left(\frac{1}{\theta_{12}}\right)^j\left(\frac{1}{\theta_{22}}\right)^{m-d-j}e^{-\frac{w_1}{\theta_{.1}}-\frac{w_2}{\theta_{.2}}}}{P(D=d,D_{11}=i,D_{12}=k)}. \quad (17)$$

Derivation of $E\left[e^{t\widehat{\theta}_{11}}|\mathcal{D}^*\right]$

$$E\left[e^{t\widehat{\theta}_{11}}|\mathcal{D}^*\right]$$
$$=\sum_{d=2}^{m-2}\sum_{k=1}^{m-d-1}\sum_{i=1}^{d-1}\left[E\left[e^{t\widehat{\theta}_{11}}|D=d,D_{11}=i,D_{12}=k\right]P(D=d,D_{11}=i,D_{12}=k|\mathcal{D}^*)\right]$$
$$=\frac{1}{P(\mathcal{D}^*)}\sum_{d=2}^{m-2}\sum_{k=1}^{m-d-1}\sum_{i=1}^{d-1}\left[E\left[e^{\frac{t}{i}w_1}|D=d,D_{11}=i,D_{12}=k\right]P(D=d,D_{11}=i,D_{12}=k)\right]$$
$$=\frac{c}{P(\mathcal{D}^*)}\sum_{d=2}^{m-2}\sum_{k=1}^{m-d-1}\sum_{i=1}^{d-1}\Bigg[\binom{d}{i}\left(\frac{\theta_{21}}{\theta_{11}+\theta_{21}}\right)^i\left(\frac{\theta_{11}}{\theta_{11}+\theta_{21}}\right)^{d-i}\binom{m-d}{k}\left(\frac{\theta_{22}}{\theta_{12}+\theta_{22}}\right)^k\times$$
$$\left(\frac{\theta_{12}}{\theta_{12}+\theta_{22}}\right)^{m-d-i}e^{-(\frac{1}{\theta_{.1}}-\frac{t}{i})\tau\left[m-d+\sum\limits_{j=d+1}^{m}R_j\right]}\int_0^{\tau}\int_0^{z_{d-1:m:n}}\dots\int_0^{z_{2:m:n}}\left(\frac{1}{\theta_{.1}}-\frac{t}{i}\right)^d\times$$

$$e^{-(\frac{1}{\theta_{.1}}-\frac{t}{i})\sum_{j=1}^{d} z_{j:m:n}(1+R_j)} \prod_{j=1}^{d} dz_{j:m:n} \Big(1-\frac{\theta_{.1}t}{i}\Big)^{-d} \Big(\frac{1}{\theta_{.1}}\Big)^{d} \Big(\frac{1}{\theta_{.2}}\Big)^{(m-d)} \int_{\tau}^{z_{m:m:n}} \ldots \times$$

$$\int_{\tau}^{z_{d+2:m:n}} e^{-\frac{1}{\theta_{.2}}\sum_{j=d+1}^{m}(z_{j:m:n}-\tau)(1+R_j)} \prod_{j=1}^{d+1} dz_{j:m:n}$$

$$= \frac{c}{P(\mathcal{D}^*)} \sum_{d=2}^{m-2} \sum_{k=1}^{m-d-1} \sum_{i=1}^{d-1} \Bigg[\binom{d}{i} \Big(\frac{\theta_{21}}{\theta_{11}+\theta_{21}}\Big)^{i} \Big(\frac{\theta_{11}}{\theta_{11}+\theta_{21}}\Big)^{d-i} \binom{m-d}{k} \Big(\frac{\theta_{22}}{\theta_{12}+\theta_{22}}\Big)^{k} \times$$

$$\Big(\frac{\theta_{12}}{\theta_{12}+\theta_{22}}\Big)^{m-d-k} \Big(1-\frac{\theta_{.1}t}{i}\Big)^{-d} \frac{e^{-(\frac{1}{\theta_{.1}}-\frac{t}{i})\tau\big[m-d+\sum_{j=d+1}^{m} R_j\big]}}{\prod_{j=1}^{m-d}\sum_{p=j}^{m-d}(1+R_p)} \times$$

$$\sum_{l=0}^{d} \frac{(-1)^l}{\big[\prod_{j=1}^{l}\sum_{p=d-l+1}^{d-l+j}(1+R_p)\big]\big[\prod_{j=1}^{d-l}\sum_{p=j}^{d-l}(1+R_p)\big]} e^{-\big(\frac{1}{\theta_{.1}}-\frac{t}{i}\big)\tau\sum_{j=d-l+1}^{d}(1+R_j)}$$

(by using Lemma 1 from [7])

$$= \frac{c}{P(\mathcal{D}^*)} \sum_{d=2}^{m-2} \sum_{i=1}^{d-1} \Bigg[\binom{d}{i} \Big(\frac{\theta_{21}}{\theta_{11}+\theta_{21}}\Big)^{i} \Big(\frac{\theta_{11}}{\theta_{11}+\theta_{21}}\Big)^{d-i} \Big[1-\Big(\frac{\theta_{22}}{\theta_{12}+\theta_{22}}\Big)^{m-d} - \Big(\frac{\theta_{12}}{\theta_{12}+\theta_{22}}\Big)^{m-d}\Big] \times$$

$$\sum_{l=0}^{d} \frac{(-1)^l\ e^{-\big(\frac{1}{\theta_{.1}}-\frac{t}{i}\big)\tau\big[l+m-d+\sum_{j=d-l+1}^{m} R_j\big]}}{\big[\prod_{j=1}^{l}\sum_{p=d-l+1}^{d-l+j}(1+R_p)\big]\big[\prod_{j=1}^{d-l}\sum_{p=j}^{d-l}(1+R_p)\big]} \ \frac{\Big(1-\frac{\theta_{.1}t}{i}\Big)^{-d}}{\prod_{j=1}^{m-d}\sum_{p=j}^{m-d}(1+R_p)} \Bigg].$$

Proof of Theorem 1.

The distribution of $\widehat{\theta}_{11}$ is obtained by inverting its MGF. Hence the distribution function of $\widehat{\theta}_{11}$ is,

$$F_{\widehat{\theta}_{11}|\mathcal{D}^*}(x) = \frac{c}{P(\mathcal{D}^*)} \sum_{d=2}^{m-2} \sum_{i=1}^{d-1} \Bigg[\binom{d}{i} \Big(\frac{\theta_{21}}{\theta_{11}+\theta_{21}}\Big)^{i} \Big(\frac{\theta_{11}}{\theta_{11}+\theta_{21}}\Big)^{d-i} \Big[1-\Big(\frac{\theta_{22}}{\theta_{12}+\theta_{22}}\Big)^{m-d} -$$

$$\Big(\frac{\theta_{12}}{\theta_{12}+\theta_{22}}\Big)^{m-d}\Big] \frac{1}{\prod_{j=1}^{m-d}\sum_{p=j}^{m-d}(1+R_p)} \times$$

$$\sum_{l=0}^{d} \Big[\frac{(-1)^l e^{-\frac{\tau}{\theta_{.1}}\big[l+m-d+\sum_{j=d-l+1}^{m} R_j\big]}}{\big[\prod_{j=1}^{l}\sum_{p=d-l+1}^{d-l+j}(1+R_p)\big]\big[\prod_{j=1}^{d-l}\sum_{p=j}^{d-l}(1+R_p)\big]} \times$$

$$F_G\Big(x; \frac{\tau}{i}\big[l+m-d+\sum_{j=d-l+1}^{m} R_j\big], d, \frac{i}{\theta_{.1}}\Big)\Big]\Bigg].$$

Similarly, the distribution function of $\widehat{\theta}_{21}$ is obtained from the expression of $F_{\widehat{\theta}_{11}|\mathcal{D}^*}(x)$ by interchanging θ_{11} and θ_{21} and is given as,

$$\begin{aligned}F_{\widehat{\theta}_{21}|\mathcal{D}^*}(x) =& \frac{c}{P(\mathcal{D}^*)}\sum_{d=2}^{m-2}\sum_{i=1}^{d-1}\Bigg[\binom{d}{i}\Big(\frac{\theta_{11}}{\theta_{11}+\theta_{21}}\Big)^{i}\Big(\frac{\theta_{21}}{\theta_{11}+\theta_{21}}\Big)^{d-i}\Bigg[1-\Big(\frac{\theta_{22}}{\theta_{12}+\theta_{22}}\Big)^{m-d} \\ & -\Big(\frac{\theta_{12}}{\theta_{12}+\theta_{22}}\Big)^{m-d}\Bigg]\frac{1}{\prod_{j=1}^{m-d}\sum_{p=j}^{m-d}(1+R_p)}\times \\ & \sum_{l=0}^{d}\Big[\frac{(-1)^l e^{-\frac{\tau}{\theta_{.1}}\big[l+m-d+\sum_{j=d-l+1}^{m}R_j\big]}}{\big[\prod_{j=1}^{l}\sum_{p=d-l+1}^{d-l+j}(1+R_p)\big]\big[\prod_{j=1}^{d-l}\sum_{p=j}^{d-l}(1+R_p)\big]}\times \\ & F_G\Big(x;\frac{\tau}{i}\big[l+m-d+\sum_{j=d-l+1}^{m}R_j\big],d,\frac{i}{\theta_{.1}}\Big)\Big]\Bigg].\end{aligned}$$

Derivation of $E\big[e^{t\widehat{\theta}_{12}}|\mathcal{D}^*\big]$

$$\begin{aligned}& E\big[e^{t\widehat{\theta}_{12}}|\mathcal{D}^*\big] \\ =& \sum_{d=2}^{m-2}\sum_{k=1}^{m-d-1}\sum_{i=1}^{d-1}\Big[E\big[e^{t\widehat{\theta}_{12}}|D=d,D_{11}=i,D_{12}=k\big]P(D=d,D_{11}=i,D_{12}=k|\mathcal{D}^*)\Big] \\ =& \frac{1}{P(\mathcal{D}^*)}\sum_{d=2}^{m-2}\sum_{k=1}^{m-d-1}\sum_{i=1}^{d-1}\Big[E\big[e^{\frac{t}{k}w_2}|D=d,D_{11}=i,D_{12}=k\big]P(D=d,D_{11}=i,D_{12}=k)\Big] \\ =& \frac{c}{P(\mathcal{D}^*)}\sum_{d=2}^{m-2}\sum_{k=1}^{m-d-1}\sum_{i=1}^{d-1}\Bigg[\binom{d}{i}\Big(\frac{\theta_{21}}{\theta_{11}+\theta_{21}}\Big)^{i}\Big(\frac{\theta_{11}}{\theta_{11}+\theta_{21}}\Big)^{d-i}\binom{m-d}{k}\Big(\frac{\theta_{22}}{\theta_{12}+\theta_{22}}\Big)^{k}\times \\ & \Big(\frac{\theta_{12}}{\theta_{12}+\theta_{22}}\Big)^{m-d-i}e^{-\frac{1}{\theta_{.1}}\tau\big[m-d+\sum_{j=d+1}^{m}R_j\big]}\Big(\frac{1}{\theta_{.1}}\Big)^{d}\int_0^{\tau}\int_0^{z_{d-1:m:n}}\dots\int_0^{z_{2:m:n}}e^{-\frac{1}{\theta_{.1}}\sum_{j=1}^{d}z_{j:m:n}(1+R_j)}\times \\ & \prod_{j=1}^{d}dz_{j:m:n}\Big(\frac{1}{\theta_{.2}}-\frac{t}{k}\Big)^{m-d}\int_{\tau}^{z_{m:m:n}}\dots\int_{\tau}^{z_{d+2:m:n}}e^{-(\frac{1}{\theta_{.2}}-\frac{t}{k})\sum_{j=d+1}^{m}(z_{j:m:n}-\tau)(1+R_j)}\prod_{j=1}^{d+1}dz_{j:m:n}\times \\ & \Big(1-\frac{t\theta_{.2}}{k}\Big)^{-(m-d)} \\ =& \frac{c}{P(\mathcal{D}^*)}\sum_{d=2}^{m-2}\sum_{k=1}^{m-d-1}\Bigg[\Big[1-\Big(\frac{\theta_{21}}{\theta_{11}+\theta_{21}}\Big)^{d}-\Big(\frac{\theta_{11}}{\theta_{11}+\theta_{21}}\Big)^{d}\Big]\binom{m-d}{k}\Big(\frac{\theta_{22}}{\theta_{12}+\theta_{22}}\Big)^{k}\times \\ & \Big(\frac{\theta_{12}}{\theta_{12}+\theta_{22}}\Big)^{m-d-k}\sum_{l=0}^{d}\frac{(-1)^l e^{-\frac{\tau}{\theta_{.1}}\big[l+m-d+\sum_{i=d-l+1}^{m}R_i\big]}}{\big[\prod_{j=1}^{l}\sum_{p=d-l+1}^{d-l+j}(1+R_p)\big]\big[\prod_{j=1}^{d-l}\sum_{p=j}^{d-l}(1+R_p)\big]}\times\end{aligned}$$

$$\frac{\left(1-\frac{t\theta_{.2}}{k}\right)^{-(m-d)}}{\prod_{j=1}^{m-d}\sum_{p=j}^{m-d}(1+R_p)}\Bigg]$$

(by using Lemma 1 from [7]).

Proof of Theorem 2

The distribution function of $\widehat{\theta}_{12}$ is obtained by inverting its MGF and is given by,

$$\begin{aligned} F_{\widehat{\theta}_{12}}(x) =& \frac{c}{P(\mathcal{D}^*)}\sum_{d=2}^{m-2}\sum_{k=1}^{m-d-1}\Bigg[\Bigg[1-\Big(\frac{\theta_{21}}{\theta_{11}+\theta_{21}}\Big)^d-\Big(\frac{\theta_{11}}{\theta_{11}+\theta_{21}}\Big)^d\Bigg]\binom{m-d}{k}\Big(\frac{\theta_{22}}{\theta_{12}+\theta_{22}}\Big)^k\times \\ & \Big(\frac{\theta_{12}}{\theta_{12}+\theta_{22}}\Big)^{m-d-k}\sum_{l=0}^{d}\frac{(-1)^l e^{-\frac{\tau}{\theta_{.1}}\left[l+m-d+\sum_{i=d-l+1}^{m}R_i\right]}}{\left[\prod_{j=1}^{l}\sum_{p=d-l+1}^{d-l+j}(1+R_p)\right]\left[\prod_{j=1}^{d-l}\sum_{p=j}^{d-l}(1+R_p)\right]}\times \\ & \frac{1}{\prod_{j=1}^{m-d}\sum_{p=j}^{m-d}(1+R_p)}\;F_G\Big(x;0,m-d,\frac{k}{\theta_{.2}}\Big)\Bigg]. \end{aligned}$$

Similarly distribution function of $\widehat{\theta}_{22}$ is obtained by interchanging θ_{12} and θ_{22} and is given by,

$$\begin{aligned} F_{\widehat{\theta}_{22}}(x) =& \frac{c}{P(\mathcal{D}^*)}\sum_{d=2}^{m-2}\sum_{k=1}^{m-d-1}\Bigg[\Bigg[1-\Big(\frac{\theta_{21}}{\theta_{11}+\theta_{21}}\Big)^d-\Big(\frac{\theta_{11}}{\theta_{11}+\theta_{21}}\Big)^d\Bigg]\binom{m-d}{k}\Big(\frac{\theta_{12}}{\theta_{12}+\theta_{22}}\Big)^k\times \\ & \Big(\frac{\theta_{22}}{\theta_{12}+\theta_{22}}\Big)^{m-d-k}\sum_{l=0}^{d}\frac{(-1)^l e^{-\frac{\tau}{\theta_{.1}}\left[l+m-d+\sum_{i=d-l+1}^{m}R_i\right]}}{\left[\prod_{j=1}^{l}\sum_{p=d-l+1}^{d-l+j}(1+R_p)\right]\left[\prod_{j=1}^{d-l}\sum_{p=j}^{d-l}(1+R_p)\right]}\times \\ & \frac{1}{\prod_{j=1}^{m-d}\sum_{p=j}^{m-d}(1+R_p)}\;F_G\Big(x;0,m-d,\frac{k}{\theta_{.2}}\Big)\Bigg]. \end{aligned}$$

Derivation of $P(\mathcal{D}^*)$

$$\begin{aligned} P(\mathcal{D}^*) =& P(D_{11}>0, D_{21}>0, D_{12}>0, D_{22}>0) \\ =& \sum_{d=2}^{m-2}\sum_{i=1}^{d-1}\sum_{k=1}^{m-d-1} P(D=d, D_{11}=i, D_{21}=k) \\ =& c\sum_{d=2}^{m-2}\sum_{i=1}^{d-1}\sum_{k=1}^{m-d-1}\binom{d}{i}\binom{m-d}{k}\Big(\frac{1}{\theta_{11}}\Big)^i\Big(\frac{1}{\theta_{21}}\Big)^{d-i}\Big(\frac{1}{\theta_{12}}\Big)^k\Big(\frac{1}{\theta_{22}}\Big)^{m-d-k}\int_V e^{-\frac{w_1}{\theta_{.1}}-\frac{w_2}{\theta_{.2}}}\times \end{aligned}$$

$$\prod_{i=1}^{m} dz_{i:m:n}$$

$$=c\sum_{d=2}^{m-2}\sum_{k=1}^{m-d-1}\sum_{i=1}^{d-1}\Bigg[\binom{d}{i}\Big(\frac{\theta_{21}}{\theta_{11}+\theta_{21}}\Big)^{i}\Big(\frac{\theta_{11}}{\theta_{11}+\theta_{21}}\Big)^{d-i}\binom{m-d}{k}\Big(\frac{\theta_{22}}{\theta_{12}+\theta_{22}}\Big)^{k}\times$$

$$\Big(\frac{\theta_{12}}{\theta_{12}+\theta_{22}}\Big)^{m-d-i} e^{-\frac{\tau}{\theta_{.1}}\left[m-d+\sum\limits_{j=d+1}^{m} R_j\right]}\Big(\frac{1}{\theta_{.1}}\Big)^{d}\int_0^{\tau}\int_0^{z_{d-1:m:n}}\ldots\int_0^{z_{2:m:n}}\times$$

$$e^{-\frac{1}{\theta_{.1}}\sum\limits_{j=1}^{d} z_{j:m:n}(1+R_j)}\prod_{j=1}^{d} dz_{j:m:n}\Big(\frac{1}{\theta_{.2}}\Big)^{(m-d)}\int_{\tau}^{z_{m:m:n}}\ldots\int_{\tau}^{z_{d+2:m:n}} e^{-\frac{1}{\theta_{.2}}\sum\limits_{j=d+1}^{m}(z_{j:m:n}-\tau)(1+R_j)}\times$$

$$\prod_{j=1}^{d+1} dz_{j:m:n}\ \{\text{where, } V=\{0<z_{1:m:n}<\ldots<z_{d:m:n}<\tau<z_{d+1:m:n}<\ldots<z_{m:m:n}\}\}$$

$$=c\sum_{d=2}^{m-2}\sum_{k=1}^{m-d-1}\sum_{i=1}^{d-1}\Bigg[\binom{d}{i}\Big(\frac{\theta_{21}}{\theta_{11}+\theta_{21}}\Big)^{i}\Big(\frac{\theta_{11}}{\theta_{11}+\theta_{21}}\Big)^{d-i}\binom{m-d}{k}\Big(\frac{\theta_{22}}{\theta_{12}+\theta_{22}}\Big)^{k}\times$$

$$\Big(\frac{\theta_{12}}{\theta_{12}+\theta_{22}}\Big)^{m-d-i}\sum_{l=0}^{d}\frac{(-1)^l e^{-\frac{\tau}{\theta_{.1}}\left[l+m-d+\sum_{j=d-l+1}^{m} R_j\right]}}{\left[\prod_{j=1}^{l}\sum_{p=d-l+1}^{d-l+j}(1+R_p)\right]\left[\prod_{j=1}^{d-l}\sum_{p=j}^{d-l}(1+R_p)\right]}\times$$

$$\frac{1}{\prod_{j=1}^{m-d}\sum_{p=j}^{m-d}(1+R_p)}\Bigg]$$

(by using Lemma 1 from [7])

$$=c\sum_{d=2}^{m-2}\Bigg[\Bigg[1-\Big(\frac{\theta_{21}}{\theta_{11}+\theta_{21}}\Big)^{d}-\Big(\frac{\theta_{11}}{\theta_{11}+\theta_{21}}\Big)^{d}\Bigg]\Bigg[1-\Big(\frac{\theta_{22}}{\theta_{12}+\theta_{22}}\Big)^{m-d}-\Big(\frac{\theta_{12}}{\theta_{12}+\theta_{22}}\Big)^{m-d}\Bigg]$$

$$\sum_{l=0}^{d}\frac{(-1)^l e^{-\frac{\tau}{\theta_{.1}}\left[l+m-d+\sum_{j=d-l+1}^{m} R_j\right]}}{\left[\prod_{j=1}^{l}\sum_{p=d-l+1}^{d-l+j}(1+R_p)\right]\left[\prod_{j=1}^{d-l}\sum_{p=j}^{d-l}(1+R_p)\right]}\frac{1}{\prod_{j=1}^{m-d}\sum_{p=j}^{m-d}(1+R_p)}\Bigg].$$

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