On Improved Accelerated Sequential Estimation of the Mean of an Inverse Gaussian Distribution

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Abstract: Hu (2020) has developed an improved accelerated sequential sampling design for the confidence interval estimation of an unknown normal mean when the variance also remains unknown. Motivated by his work, in this paper, we propose improved accelerated sequential procedures to estimate the unknown mean μ of an inverse Gaussian distribution, when the scale parameter λ also remains unknown. We consider the problems of minimum risk and bounded risk point estimation, under a weighted squared-error loss function. We aim at controlling the associated risk functions and second-order asymptotics are also obtained. We establish the superiority of this improved accelerated sequential procedure over the Hall's (1983) accelerated sequential procedure in estimating an inverse Gaussian mean. Appropriate simulations and real data examples are provided in support of the encouraging performances of our proposed methodologies.

Keywords: Bounded Risk; Improved Accelerated Sequential Procedure; Inverse Gaussian Distribution; Minimum Risk; Risk Per Unit Cost; Second-Order Asymptotics; Weighted Squared-Error Loss.

Mathematics Subject Classifications: 62L12, 62L05, 62F12, 62P30

1. Introduction

Dantzig (1940) proved the non-existence of any fixed sample size procedure to construct the confidence interval of preassigned width and coverage probability for a normal mean in the ignorance of any knowledge about the variance. In order to deal with this estimation problem, Stein (1945) proposed a two-stage procedure. Stein's two-stage procedure is easy to operate as it requires only two-stages and achieves the exact coverage probability but the difference between average sample number and the 'optimal' fixed sample size does not remain asymptotically bounded.

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As a result, the two-stage procedure leads us to considerable oversampling. In order to reduce the problem of oversampling, Anscombe (1953) and Chow and Robbins (1965) have

developed the theory of sequential estimation.

Starr (1966) and Woodroofe (1977) adopted the sequential estimation strategies for the confidence interval estimation problem discussed above. However, the purely sequential procedures are complicated in nature to apply and achieve the target coverage probability only asymptotically. In order to combine the advantages of two-stage and purely sequential procedures, Hall (1981) developed a three-stage procedure in order to construct fixed-width confidence interval for a normal mean with unknown variance. Hall (1981), through his theoretical and numerical findings, showed that if another stage is appended to Stein's two-stage procedure, the resulting three-stage procedure becomes strongly competitive to both the two-stage and purely sequential procedures. Later on, for the same reasons behind the use of a three-stage procedure, Hall (1983) developed an 'accelerated' sequential procedure, in which the number of stages can be reduced by a predetermined factor at the cost of only a finite number of observations. Mukhopadhyay (1996) provided an alternative formulation of accelerated sequential procedures with applications to parametric and nonparametric estimation

Recently Hu (2020) proposed an improved accelerated sequential sampling design to handle the problem of fixed-width confidence interval estimation of an unknown normal mean when the variance is assumed unknown. He established that his procedure substantially saves sampling operations while preserving efficiency and consistency properties. In the present paper, we extend the idea of Hu (2020) to handle the point estimation problems related to the mean μ of an inverse Gaussian distribution, when the scale parameter λ is assumed to be unknown. We consider the problems of minimum risk and bounded risk point estimation respectively and present the improved accelerated sequential sampling strategies to handle these problems. We obtain the associated second-order asymptotics as well.

1.1. The Inverse Gaussian Distribution

Let $X_1, X_2, ..., X_n$ be an independent and identically distributed (iid) sequence of inverse Gaussian random variables with a common density function:

$$f(x;\mu,\lambda) = \left\{\frac{\lambda}{2\pi x^3}\right\}^{1/2} \exp\left\{\frac{-\lambda\,(x-\mu)^2}{2x\mu^2}\right\}, x > 0, \tag{1.1}$$

where $\mu > 0$ and $\lambda > 0$ are unknown mean and scale parameters respectively.

Moreover, for a random sample $X_1, X_2, ..., X_n$ of fixed size $n(\geq 2)$ from (1.1), the estimators of μ and $1/\lambda$ are given by $\overline{X}_n$ (sample mean) and $1/\widehat{\lambda}_n$, respectively, where,

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \tag{1.2}$$

$$\frac{1}{\widehat{\lambda}_n} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{X_i} - \frac{1}{\overline{X}_n}\right). \tag{1.3}$$

where $1/\widehat{\lambda}_n$ is the maximum likelihood estimator (MLE) of $1/\lambda$. Further, we have:

$$\frac{n\lambda}{\widehat{\lambda}_n} \sim \chi^2_{n-1}. \tag{1.4}$$

An inverse Gaussian distribution given in (1.1) was first developed by Schrodinger (1915) to model the first passage time of a Brownian motion with positive drift. Over many years, it is well considered as an alternative to a Gaussian model with many common inferential properties. It is also used for modeling positively skewed data, apart from the well known models such as Pareto, gamma, lognormal, folded normal etc.

This distribution possesses a wide range of applications in real life. One can use an inverse Gaussian model in actuarial science for claim cost analysis, in hydrology for analysis of floods, in meteorology to model the distribution of wind speed or in physiology to study the blood circulation and body fluids. For more details regarding the applications of an inverse Gaussian distribution, one may refer to Folks and Chhikara (1978), Chhikara and Folks (1989), Seshadri (1993, 1999) and Johnson et al. (1994). An R package has been developed by Leiva et al. (2008) for a general class of inverse Gaussian models.

1.2. A Brief Review of the Recent Developments

The problem of estimating an inverse Gaussian mean has been considered widely in the literature of sequential analysis. Chaturvedi (1985) developed a sequential procedure for estimating an inverse Gaussian mean with prescribed proportional closeness and established that his procedure is 'asymptotically efficient' and 'asymptotically consistent' in the sense of Chow and Robbins (1965). He also discussed the associated second-order properties in an addendum to his 1985 paper [See Chaturvedi (1986)]. An asymptotically efficient sequential estimation of an inverse Gaussian mean has been considered by Chaturvedi et al. (1991). Bapat (2018) developed the purely sequential procedure for the bounded risk point estimation of an inverse gaussian mean under an interesting weighted squared-error loss function and discussed the associated first and second-order asymptotics.

Chaturvedi et al. (2019) have developed two-stage, three-stage and accelerated sequential procedures for minimum risk and bounded risk point estimation of an inverse Gaussian mean and established the associated second-order asymptotics. Moreover, Chaturvedi et al. (2020a) have developed purely sequential minimum risk point estimation methodologies for the mean μ and scale parameter λ of an inverse Gaussian distribution and established the associated first and second-order asymptotics. They have proposed an improved estimator for estimating the scale parameter λ as well. In another paper, they have developed purely sequential and k-stage procedures for estimating an inverse Gaussian mean with prescribed proportional closeness and discussed the associated second-order asymptotics [See Chaturvedi et al. (2020b)], where, they have also worked under a family of loss functions.

1.3. Outline of the Paper

In Section 2, we give the setup of our minimum risk and bounded risk point estimation problems. We establish the non-existence of any fixed-sample size procedures to deal with these problems. In Section 3, we propose the improved accelerated sequential procedures [Hu (2020)] for our estimation problems. We obtain the associated second-order asymptotics for the expected sample size and risk as well. In Section 4, we provide simulations in support of the accuracy of our proposed procedures. Section 5 of our paper presents illustrations using a real data set and we end with a brief set of conclusions in Section 6. Proofs of some important results are provided in the Appendix.

2. The Estimation Problems

The problems of minimum risk and bounded risk point estimation are explained in the Sections 2.1 and 2.2 respectively.

2.1. Minimum Risk Point Estimation Problem

Suppose the loss incurred in estimating μ by $\overline{X}_n$ be:

$$L(\mu, \overline{X}_n) = \frac{A\left(\overline{X}_n - \mu\right)^2}{\mu^3} + cn\lambda^k, \tag{2.1}$$

where A, c and k are fixed known positive constants. This loss function is of the form, "weighted squared error plus cost". One may refer to [Chaturvedi et al. (2020a)] for a detailed explanation of picking the loss function given in (2.1).

The expression of the risk $R_n(c)$ corresponding to the loss function (2.1) is given by:

$$R_n(c) = E[L(\mu, \overline{X}_n)] = \frac{A}{n\lambda} + cn\lambda^k. \tag{2.2}$$

We wish to minimize the risk function given in (2.2). The value '$n = n_0$' minimizing this risk function is given by:

$$n_0 = \left(\frac{A}{c}\right)^{1/2} \left(\frac{1}{\lambda}\right)^{\frac{k+1}{2}}. \tag{2.3}$$

and the corresponding minimum risk is:

$$R_{n_0}(c) = 2c\lambda^k n_0. \tag{2.4}$$

However, it is clear from (2.3) that, in the absence of any knowledge about 'λ', no fixed sample size procedure can minimize the risk (2.2) for all values of 'λ'. In order to deal with this problem, we resort in developing an improved accelerated sequential sampling strategy suggested by Hu (2020). The proposed procedure can be found in the Section 3.1.

2.2. Bounded Risk Point Estimation Problem

Suppose the loss incurred in estimating μ by $\overline{X}_n$ be:

$$L(\mu, \overline{X}_n) = \frac{\left(\overline{X}_n - \mu\right)^2}{\mu^3}. \tag{2.5}$$

One may obtain the expression of the risk corresponding to the loss function (2.5) as follows:

$$R_n = E[L(\mu, \overline{X}_n)] = \frac{1}{n\lambda}. \tag{2.6}$$

Further, we consider a cost function $C_n(> 0)$ as:

$$C_n = cn\lambda^k, \tag{2.7}$$

where c and k are fixed and known positive constants. From (2.6) and (2.7), one may obtain the "risk per unit cost", denoted by $RPUC_n$ as:

$$RPUC_n = \frac{R_n}{C_n} = \frac{1}{cn^2\lambda^{k+1}}. \tag{2.8}$$

Now we wish to bound the "risk per unit cost" given in (2.8) by a fixed and known positive constant '$w(> 0)$' and thus require that, '$RPUC_n \leq w$' for all μ, λ. This leads us to determine the required "optimal fixed sample size", had 'λ' been known as:

$$n^* = \frac{1}{(cw)^{1/2}} \left(\frac{1}{\lambda}\right)^{(k+1)/2}. \tag{2.9}$$

However, from (2.9), it is clear that, when 'λ' is unknown, no fixed sample size procedure yields '$RPUC_n \leq w$' for all λ. Hence, we resort in developing the improved accelerated sequential strategy to solve this problem as well. The proposed procedure can be found in the Section 3.2.

3. The Improved Accelerated Sequential Sampling Strategies

We now discuss our methodologies to handle the estimation problems described under Section 2.

3.1. Estimation of the Mean for Minimum Risk Problem

We start with the pilot sample $X_1, ..., X_m$ of size $m(\geq 2)$ from the distribution (1.1). Fixing $0 \leq \rho \leq 1$, a proportion and $p \geq 1$, an integer, we record 'p' additional observations at-a-time successively until we terminate sampling according to the following stopping rule:

$$L_1 = \inf\left\{ n \geq 0 : m + pn \geq \rho \left(\frac{A}{c}\right)^{1/2} \left(\frac{1}{\widehat{\lambda}_{m+pn}}\right)^{(k+1)/2} \right\}. \tag{3.1}$$

where $1/\widehat{\lambda}$ stands for the MLE of $1/\lambda$. The stopping time L_1 indicates the number of sampling operations needed in the sequential sampling stage under a specific combination of A, c, m, ρ, k and p values. Hence, a preliminary sample size is given by:

$$T_1 = m + pL_1. \tag{3.2}$$

Now we define the terminated sample size as follows:

$$N_1 = \max\left\{T_1, \left\lfloor \left(\frac{A}{c}\right)^{1/2} \left(\frac{1}{\widehat{\lambda}_{T_1}}\right)^{(k+1)/2} + \epsilon \right\rfloor + 1\right\}. \tag{3.3}$$

where $\lfloor u \rfloor$ denotes the largest integer smaller than u and ϵ is a non-negative constant, which depends on the conscience of an experimenter. The stopping rule 'N_1' estimates 'n_0' from (2.3). Upon termination, we have a final data set $X_1, X_2, ..., X_m, ..., X_{T_1}, ..., X_{N_1}$ and we now estimate the mean μ by $\overline{X}_{N_1}$. Further, we assume that the following conditions are satisfied:

$$m = pm_0 + 1, \ \ where, \ \ m_0 > 2/p, \ \ and \ \ m/n_0 < \rho. \tag{3.4}$$

Remark 3.1. *Note that, for $p = 1$ and $0 < \rho < 1$, the proposed procedure is similar with the Hall's accelerated sequential procedure in estimating an inverse Gaussian mean for the minimum risk problem [See Chaturvedi et al. (2019)]. Moreover, for $0 < \rho < 1$ and $p \geq 2$ simultaneously, our proposed procedure can save sampling operations by taking multiple observations instead of one at-a-time in the sequential sampling stage.*

Second-Order Asymptotics

Now we provide the second-order asymptotics for the average sample number, $E(N_1)$ and the risk, $R_{N_1}(c)$ associated with our improved accelerated sequential sampling procedure defined in (3.1)-(3.3). These results are given in the following theorem:

Theorem 3.1. *For the accelerated sequential procedure defined in Equations (3.1)-(3.3), for all fixed μ, λ, predetermined $0 < \rho \leq 1, p \geq 1$ and under the assumption (3.4), we have as $c \to 0$:*

$$E(N_1) = n_0 - \frac{(k+1)(k+3)}{4\rho} + \epsilon + \frac{1}{2} + o(1), \tag{3.5}$$

$$R_{N_1}(c) = R_{n_0}(c) + \frac{c\lambda^k}{2p}(k+1)^2 + o(c^{-1/2}). \tag{3.6}$$

3.2. Estimation of the Mean for Bounded Risk Problem

Again we start with the pilot sample $X_1, ..., X_m$ of size $m(\geq 2)$ from the distribution (1.1). Fixing $0 \leq \eta \leq 1$, a proportion and $q \geq 1$, an integer, we record 'q' additional observations at-a-time successively until we terminate sampling according to the following stopping rule:

$$L_2 = \inf\left\{n \geq 0 : m + qn \geq \eta \left(\frac{1}{cw}\right)^{1/2} \left(\frac{1}{\widehat{\lambda}_{m+qn}}\right)^{(k+1)/2}\right\}. \tag{3.7}$$

The stopping time L_2 indicates the number of sampling operations needed in the sequential sampling stage under a specific combination of c, m, η, k and q values. Hence, a preliminary sample size is given by:

$$T_2 = m + qL_2. \tag{3.8}$$

Now we define the terminated sample size as follows:

$$N_2 = \max\left\{T_2, \left\lfloor \left(\frac{1}{cw}\right)^{1/2} \left(\frac{1}{\widehat{\lambda}_{T_2}}\right)^{(k+1)/2} + \epsilon \right\rfloor + 1\right\}. \tag{3.9}$$

The stopping rule 'N_2' estimates 'n^*' from (2.9). Upon termination, we have a final data set $X_1, X_2, ..., X_m, ..., X_{T_2}, ..., X_{N_2}$ and we now estimate the mean μ by $\overline{X}_{N_2}$. Further, we assume that the following conditions are satisfied:

$$m = qm^* + 1, \ \textit{where}, \ \ m^* > 2/q, \ \textit{and} \ \ m/n^* < \eta. \tag{3.10}$$

Remark 3.2. *Note that, for $q = 1$ and $0 < \eta < 1$, the proposed procedure (3.7)-(3.9) is similar with the Hall's accelerated sequential procedure in estimating an inverse Gaussian mean for the bounded risk problem [See Chaturvedi et al. (2019)]. Moreover, for $0 < \eta < 1$ and $q \geq 2$ simultaneously, our proposed procedure can save sampling operations by taking multiple observations instead of one at-a-time in the sequential sampling stage.*

Second-Order Asymptotics

Now we provide second-order asymptotics for the average sample number, $E(N_2)$ and the risk per unit cost, $R_{N_2}(c)$ associated with our improved accelerated sequential sampling procedure defined in (3.7)-(3.9). These results are given in the following theorem:

Theorem 3.2. *For the accelerated sequential procedure defined in Equations (3.7)-(3.9), for all fixed μ, λ, predetermined $0 < \eta \leq 1, q \geq 1$ and under the assumption (3.10), we have as $\omega \to 0$:*

$$E(N_2) = n^* - \frac{(k+1)(k+3)}{4\eta} + \epsilon + \frac{1}{2} + o(1), \tag{3.11}$$

$$R_{N_2}(c) = w - \frac{w}{n^*}\left[2\left(\epsilon + \frac{1}{2}\right) - \frac{(k+1)(k+3)}{2\eta} - \frac{3(k+1)^2}{2q}\right] + o(c^{-1/2}). \tag{3.12}$$

4. Data Analysis: Simulations

4.1. Simulations for Section 3.1

We now present simulation results for the improved accelerated sequential procedure given in (3.1)-(3.3). These are summarized in Table 1. We first generated a set of pseudorandom observations at-a-time from the distribution given in (1.1). Each row in Table 1 corresponds to results from 10,000 replications where results are tabulated for different

combinations of $\mu, \lambda, k, A, m_0, \epsilon$ and n_0, where n_0 comes from (2.3). We also fix $\rho = 0.5$ throughout, just to be consistent. We further fix values of $p = 1, 3, 5$. As per Remark 3.1, $p = 1$ becomes similar to Hall's accelerated sequential procedure and we will hence compare it with results obtained using the other two values for p. These three groups can be seen as different chunks in Table 1. Lastly, we also ensure that the constraint (3.4) is met in all the scenarios.

Each block in Table 1 shows n_0 (column 1), c (column 2), the values $\bar{x}$ along with their standard errors $s_{\bar{x}}$ (column 3), the values $\overline{n}, s_{\overline{n}}$ (column 4), the ratio $\overline{n}/n_0$ (column 5), ξ_1 (column 6), where ξ_1 denotes the expansion as per of (3.5) and we expect $\bar{n}$ to be close to ξ_1, the values $\overline{z}$ (column 7) where $\bar{z}$ denotes the average risk, ξ_2 (column 8), where ξ_2 denotes the expansion as per (3.6) and one should expect $\bar{z}$ to be close to ξ_2 and ψ (column 9), where ψ denotes the number of sampling operations needed for that case.

One can note that the values of $\overline{n}$ seem to estimate n_0 reasonably well across the rows, which become closer for larger values of n_0, indicating a superior first-order efficiency. The $\overline{x}$ values are also seen to be very good at estimating the true mean of 5 in each case. The second-order efficiencies of the sample size N_1 and the risk function are also seen to be accurate on observing values of ξ_1 and ξ_2. Hence, one can conclude that all three chunks ($p = 1, 3, 5$) perform equally well in terms of efficiencies. However one may clearly note that the number of sampling operations needed reduce, as p increases. This suggests that one may control the value of p to get optimum results by only taking a handful of sample points at the sequential stage, and this in particular is an improvement over the existing Hall's accelerated sequential procedure.

Table 1. Simulation results from 10000 replications of the improved accelerated sequential methodology (3.1)-(3.3) with $m_0 = 8$, $k = 3$, $A = 0.1$

n_0	c	$\overline{x}, s_{\bar{x}}$	$\bar{n}, s_{\bar{n}}$	$\bar{n}/n_0$	ξ_1	$\bar{z}$	ξ_2	ψ
$\mu = 5,\ \lambda = 0.8,\ \rho = 0.5,\ p = 1$								
100	2.44×10^{-5}	$5.04, 0.04$	$115.28, 2.19$	1.15	110.5	0.0026	0.0028	30.03
350	1.99×10^{-6}	$4.97, 0.02$	$360.22, 2.88$	1.03	360.5	0.0007	0.0007	153.76
500	9.76×10^{-7}	$4.97, 0.01$	$511.46, 3.88$	1.02	510.5	0.0005	0.0005	231.52
$\mu = 5,\ \lambda = 0.8,\ \rho = 0.5,\ p = 3$								
100	2.44×10^{-5}	$4.97, 0.04$	$110.37, 1.95$	1.10	110.5	0.0024	0.0026	11.01
350	1.99×10^{-6}	$5.01, 0.02$	$365.84, 3.38$	1.04	360.5	0.0007	0.0007	52.29
500	9.76×10^{-7}	$5.02, 0.01$	$512.91, 3.87$	1.02	510.5	0.0005	0.0005	77.66
$\mu = 5,\ \lambda = 0.8,\ \rho = 0.5,\ p = 5$								
100	2.44×10^{-5}	$5.00, 0.04$	$111.11, 1.81$	1.11	110.5	0.0025	0.0028	6.94
350	1.99×10^{-6}	$4.99, 0.02$	$364.20, 2.86$	1.04	360.5	0.0007	0.0007	31.86
500	9.76×10^{-7}	$5.00, 0.02$	$511.69, 2.99$	1.02	510.5	0.0005	0.0005	47.17

4.2. Simulations for Section 3.2

We now present simulation results for the improved accelerated sequential procedure given in (3.7)-(3.9). These are summarized in Table 2. We first generated a set of pseudorandom observations at-a-time from the distribution given in (1.1). Each row in Table 2 corresponds to results from 10,000 replications where results are tabulated for different combinations of $\mu, \lambda, k, A, m^*, \epsilon$ and n^*, where n^* comes from (2.9). We also fix $\eta = 0.5$ throughout, just to be consistent. We further fix values of $q = 1, 3, 5$. As per Remark 3.2, $q = 1$ becomes similar to Hall's accelerated sequential procedure and we will hence compare it with results obtained using the other two values for q. These three groups can be seen as different chunks in Table 2. Lastly, we also ensure that the constraint (3.10) is met in all the scenarios.

Each block in Table 2 shows n^* (column 1), ω (column 2), the values $\bar{x}$ along with their standard errors $s_{\bar{x}}$ (column 3), the values $\overline{n}, s_{\overline{n}}$ (column 4), the ratio $\overline{n}/n^*$ (column 5), ξ_3 (column 6), where ξ_3 denotes the expansion as per of (3.11) and we expect $\bar{n}$ to be close to ξ_3, the values $\overline{z}$ (column 7) where $\bar{z}$ denotes the average risk, ξ_4 (column 8), where ξ_4 denotes the expansion as per (3.12) and one should expect $\bar{z}$ to be close to ξ_4 and ψ (column 9), where ψ denotes the number of sampling operations needed for that case.

One can note that the values of $\overline{n}$ seem to estimate n^* reasonably well across the rows, which become closer for larger values of n^*, indicating a superior first-order efficiency. The $\overline{x}$ values are also seen to be very good at estimating the true mean of 5 in each case. The second-order efficiencies of the sample size N_2 and the risk function are also seen to be accurate on observing values of ξ_3 and ξ_4. Similar to Table 1, all three chunks ($q = 1, 3, 5$) perform equally well in terms of efficiencies. However one may clearly note that the number of sampling operations needed reduce, as q increases. This suggests that one may control the value of q to get optimum results by only taking a handful of sample points at the sequential stage, and this in particular is an improvement over the existing Hall's accelerated procedure.

Table 2. Simulation results from 10000 replications of the improved accelerated sequential methodology (3.7)-(3.9) with $m^* = 8$, $k = 3$, $c = 0.1$

n^*	ω	$\overline{x}, s_{\bar{x}}$	$\bar{n}, s_{\bar{n}}$	$\bar{n}/n^*$	ξ_3	$\bar{z}$	ξ_4	ψ
$\mu = 5,\ \lambda = 0.8,\ \eta = 0.5,\ q = 1$								
100	0.0024	5.03, 0.04	100.47, 2.05	1.00	100.5	0.0029	0.0117	30.72
350	0.0001	4.99, 0.02	356.17, 4.08	1.01	350.5	0.0002	0.0002	154.45
500	9.76×10^{-5}	5.01, 0.01	497.65, 3.22	0.99	500.5	0.0001	0.0001	231.67
$\mu = 5,\ \lambda = 0.8,\ \eta = 0.5,\ q = 3$								
100	0.0024	4.98, 0.04	102.97, 2.10	1.03	100.5	0.0025	0.0087	10.78
350	0.0001	5.00, 0.02	349.37, 2.87	0.99	350.5	0.0002	0.0005	52.93
500	9.76×10^{-5}	4.99, 0.01	499.75, 3.10	0.99	500.5	0.0001	0.0001	77.94
$\mu = 5,\ \lambda = 0.8,\ \eta = 0.5,\ q = 5$								
100	0.0024	4.96, 0.04	99.40, 1.82	0.99	100.5	0.0025	0.0092	6.88
350	0.0001	4.99, 0.02	351.03, 2.68	1.00	350.5	0.0002	0.0004	32.23
500	9.76×10^{-5}	5.00, 0.01	499.69, 3.76	0.99	500.5	0.0001	0.0002	47.52

5. Analysis from Real Data

We will now apply our proposed improved accelerated sequential procedures on a real data set from reliability studies. This data set was reported by Birnbaum and Saunders (1958, 1969) and corresponds to the fatigue life, measured in cycles of 101 aluminum coupons of type 6061-T6. These aluminum sheets were cut parallel to the direction of rolling and oscillating at 18 cycles per second. These were then exposed to pressure at a stress level of 31,000 psi (pounds per square inch). All coupons were tested until they failed. The full data consists of fatigue life of 101 aluminum coupons. An inverse Gaussian model fits very well to the data with a p-value of 0.4603. One may refer to Leiva et al. (2008) and more recently to Bapat (2018) or Chaturvedi et al. (2019) for more details. Treating these data as the universe, the maximum likelihood estimates were $\widehat{\mu} = 133.73$ and $\widehat{\lambda} = 4587.16$. Our proposed sequential procedures were then applied to draw observations from the full set of data as needed. One may note that all the results are from a single run.

5.1. Real Data Analysis for Section 3.1

We first analyze the minimum risk improved accelerated sequential procedure (3.1)-(3.3). We fixed the values of $A = 50, m_0 = 5, k = 2$ and $\rho = 0.5$. Table 3 gives results for two different values of p being 1 and 3 respectively. One can note that all the $\overline{x}$ values are close to the actual mean of 133.73 and the observed n values are also close to the corresponding n_0 values. Further, the risk ratios (column 7) are also seen to be close to the corresponding ξ values (column 8). Lastly, column 9 gives the number of sampling operations needed for each scenario. One can clearly note that the values of ψ decrease, with increasing p. Hence one can control the value of p and yet get an improvement over the usual Hall's accelerated sequential procedure, in terms of collecting lesser number of samples at the sequential stage.

Table 3. Analysis of fatigue life data using improved accelerated sequential procedure (3.1)-(3.3) with $m_0 = 5, A = 50, k = 2, \rho = 0.5$ $\widehat{\mu} = 133.73$, $\widehat{\lambda} = 4587.15$

p	n_0	c	$\overline{x}$	n	n/n_0	z	ξ	ψ
1	50	2.07×10^{-13}	136.19	47	0.94	0.0004	0.0006	27
	70	1.05×10^{-13}	133.46	73	1.04	0.0003	0.0004	18
	80	8.09×10^{-14}	133.39	79	0.98	0.0002	0.0002	53
3	50	2.07×10^{-13}	131.96	53	1.06	0.0004	0.0004	10
	70	1.05×10^{-13}	133.75	72	1.02	0.0003	0.0004	14
	80	8.09×10^{-14}	133.14	86	1.07	0.0002	0.0003	18

5.2. Real Data Analysis for Section 3.2

We now analyze the bounded risk improved accelerated sequential procedure (3.7)-(3.9). We fixed the values of $c = 0.01, m_0 = 5, k = 2$ and $\eta = 0.5$. Table 4 gives results for two different values of q being 1 and 3 respectively. One can note that all the $\overline{x}$ values are close to the actual mean of 133.73 and the observed n values are also close to the corresponding

n^* values. Further, the risk ratios (column 7) are also seen to be close to the corresponding ξ values (column 8). Lastly, column 9 gives the number of sampling operations needed for each scenario. One can clearly note that the values of ψ decrease, with increasing q. Hence one can control the value of q and yet get an improvement over the usual Hall's accelerated sequential procedure, in terms of collecting lesser number of samples at the sequential stage.

Table 4. Analysis of fatigue life data using improved accelerated sequential procedure (3.7)-(3.9) with $m_0 = 5, c = 0.01, k = 2, \eta = 0.5$ $\widehat{\mu} = 133.73$, $\widehat{\lambda} = 4587.15$

q	n^*	ω	$\overline{x}$	n	n/n^*	z	ξ	ψ
1	50	4.14×10^{-13}	133.73	52	1.04	3.83×10^{-13}	5.09×10^{-13}	24
	70	2.11×10^{-13}	133.57	78	1.11	1.70×10^{-13}	2.46×10^{-13}	21
	80	1.61×10^{-13}	137.29	74	0.92	1.89×10^{-13}	1.85×10^{-13}	38
3	50	4.14×10^{-13}	134.31	59	1.18	2.97×10^{-13}	4.35×10^{-13}	6
	70	2.11×10^{-13}	131.78	66	0.94	2.37×10^{-13}	2.18×10^{-13}	10
	80	1.61×10^{-13}	131.96	76	0.95	1.79×10^{-13}	1.66×10^{-13}	16

6. Concluding Remarks

We have developed improved accelerated sequential procedures for the minimum risk and bounded risk point estimation of the mean μ of an inverse Gaussian distribution assuming the scale parameter λ is unknown. Our estimation procedures enjoy first and second-order efficiency properties as well. Moreover, our improved accelerated sequential procedure can reduce the sampling operations substantially, hence it is superior to the Hall's accelerated sequential procedure in estimating an inverse Gaussian mean. Extensive simulation analyses are carried out to support our claims and a real data example from reliability studies is also presented.

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Appendix

Proof of Theorem 3.1. Denoting by,

$$\psi_{T_1} = 1 - \left\{ \left(\frac{A}{c}\right)^{1/2} \left(\frac{1}{\widehat{\lambda}_{T_1}}\right)^{(k+1)/2} + \epsilon - \left\lfloor \left(\frac{A}{c}\right)^{1/2} \left(\frac{1}{\widehat{\lambda}_{T_1}}\right)^{(k+1)/2} + \epsilon \right\rfloor \right\},$$

we can write,

$$E(N_1) = I + II, \tag{A.1}$$

where,

$$I = E\left\{ (T_1) I\left(T_1 \geq \left\lfloor \left(\frac{A}{c}\right)^{1/2} \left(\frac{1}{\widehat{\lambda}_{T_1}}\right)^{(k+1)/2} + \epsilon \right\rfloor + 1 \right) \right\},$$

and

$$II = \left(\frac{A}{c}\right)^{1/2} E\left\{ \left(\frac{1}{\widehat{\lambda}_{T_1}}\right)^{(k+1)/2} \right\} + \epsilon + E(\psi_{T_1}). \tag{A.2}$$

It follows from Hall (1983) that, as $c \to 0$, $I = o(1)$ and ψ_{T_1} is uniform over (0,1). Thus, we obtain from (A.1) and (A.2) that,

$$E(N_1) = \left(\frac{A}{c}\right)^{1/2} E\left\{ \left(\frac{1}{\widehat{\lambda}_{T_1}}\right)^{(k+1)/2} \right\} + \epsilon + \frac{1}{2} + o(1). \tag{A.3}$$

Now we first evaluate $E\left\{ \left(1/\widehat{\lambda}_{T_1}\right)^{(k+1)/2} \right\}$. To this end, under the condition (3.4), we can write (3.1) as,

$$L_1 = \inf\Bigg\{ n \geq 0 : p(m_0+n)(p(m_0+n)+1) \geq \rho \left(\frac{A}{c}\right)^{1/2} \left(\frac{1}{\lambda}\right)^{(k+1)/2} \times p(m_0+n) \left(\frac{\lambda}{\widehat{\lambda}_{p(m_0+n)+1}}\right)^{(k+1)/2} \Bigg\},$$

which can further be written as,

$$L_1 = \inf\left\{ n \geq m_0 : pn(pn+1) \geq \rho n_0(pn) \left(\frac{\lambda}{\widehat{\lambda}_{pn+1}} \right)^{(k+1)/2} \right\}. \tag{A.4}$$

Further, we have,

$$\frac{pn\lambda}{\widehat{\lambda}_{pn+1}} \sim \chi^2_{pn}. \tag{A.5}$$

(A.5) can be expressed as the summation of 'n' *iid* chi-squared random variables with 'p' degrees of freedom, i.e., $\sum_{j=1}^{n} Z_j$, where $Z_j, j = 1, 2, ..., n$ are *iid* χ^2_p random variables.

Using Equation (A.5), we rewrite (A.4) as,

$$L_1^* = \inf\left\{ n \geq m_0 : \sum_{j=1}^{n} Z_j \leq \frac{(np)^{(k+3)/(k+1)}}{(\rho n_0)^{2/(k+1)}} \left(1 + \frac{1}{np} \right)^{2/(k+1)} \right\}, \tag{A.6}$$

where the stopping time $L_1^* = L_1 + m_0$.

Comparing (A.6) with equation (1.1) of Woodroofe (1977), we obtain in his notations:

$$S_n = \sum_{j=1}^{n} Z_j, c = \frac{p^{(k+3)/(k+1)}}{(\rho n_0)^{2/(k+1)}}, \alpha = \frac{k+3}{k+1}, \beta = \frac{k+1}{2}, \lambda = \frac{\rho n_0}{p},$$

$$L_n = \left\{ 1 + \frac{1}{np} \right\}^{2/(k+1)}, L_0 = \frac{2}{p(k+1)}, \mu = p, \tau^2 = 2p.$$

It now follows from his Theorem 2.4 that, for all $m > (k+1)/2p$, as $c \to 0$,

$$E(L_1^*) = (\rho n_0)p^{-1} + \left(\frac{k+1}{2} \right) \nu p^{-1} - \left\{ 1 + \frac{(k+1)(k+3)}{4} \right\} p^{-1} + o(1). \tag{A.7}$$

provided that $m_0 > 2/p$, where 'ν' is specified. By noting that, $T_1 = pL_1^* + 1$ with probability 1, it follows from (A.7) that,

$$E(T_1) = \rho n_0 + \left(\frac{k+1}{2} \right) \nu - \frac{(k+1)(k+3)}{4} + o(1). \tag{A.8}$$

Now, let us consider the difference,

$$D_a = T_1 - \rho \left(\frac{A}{c} \right)^{1/2} \left(\frac{1}{\widehat{\lambda}_{T_1}} \right)^{(k+1)/2}. \tag{A.9}$$

It follows from Woodroofe (1977) that, mean of the asymptotic distribution of D_a is $(k+1)\nu/2$. Now from (A.8) and (A.9), we have,

$$\left(\frac{A}{c}\right)^{1/2} E\left\{\left(\frac{1}{\widehat{\lambda}_{T_1}}\right)^{(k+1)/2}\right\} = n_0 - \frac{(k+1)(k+3)}{4\rho} + o(1). \tag{A.10}$$

Now (3.5) follows by putting (A.10) in (A.3). In order to obtain (3.6), we proceed as follows:

The risk function associated with our stopping rule 'N_1' is given by:

$$R_{N_1}(c) = c\lambda^k n_0 \left\{E\left(\frac{n_0}{N_1}\right) + E\left(\frac{N_1}{n_0}\right)\right\}. \tag{A.11}$$

Thus, we have to obtain the expression for $E(n_0/N_1)$. Applying Taylor's series expansion, we get:

$$E\left(\frac{n_0}{N_1}\right) = 1 - \frac{1}{n_0}E(N_1 - n_0) + \frac{1}{n_0^2}E(N_1 - n_0)^2 U^{-3}, \tag{A.12}$$

where, $\mid U-1 \mid \leq \mid (N_1/n_0) - 1 \mid$. Moreover, we have from Ghosh and Mukhopadhyay (1975),

$$\frac{N_1 - n_0}{n_0^{1/2}} \longrightarrow N\left(0, \frac{(k+1)^2}{2p}\right). \tag{A.13}$$

and 'U' is uniformly integrable [See Lemma 3.4 in Chaturvedi et al. (2019)]. Now using (3.5) and (A.13), we have from (A.12),

$$E\left(\frac{n_0}{N_1}\right) = 1 + \frac{(k+1)(k+3)}{4\rho n_0} - \frac{(\epsilon + 1/2)}{n_0} + \frac{(k+1)^2}{2pn_0} + o(c^{-1/2}). \tag{A.14}$$

Moreover, we have from (3.5),

$$E\left(\frac{N_1}{n_0}\right) = 1 - \frac{(k+1)(k+3)}{4\rho n_0} + \frac{(\epsilon + 1/2)}{2n_0} + o(1). \tag{A.15}$$

(3.6) now follows by putting (A.14) and (A.15) in (A.11). Proof of Theorem 3.1 is now complete.

Proof of Theorem 3.2. Denoting by,

$$\psi_{T_2} = 1 - \left\{\left(\frac{1}{cw}\right)^{1/2}\left(\frac{1}{\widehat{\lambda}_{T_2}}\right)^{(k+1)/2} + \epsilon - \left\lfloor \left(\frac{1}{cw}\right)^{1/2}\left(\frac{1}{\widehat{\lambda}_{T_2}}\right)^{(k+1)/2} + \epsilon \right\rfloor\right\},$$

we can write,

$$E(N_2) = I + II, \tag{A.16}$$

where,

$$I = E\left\{(T_2)I\left(T_2 \geq \left\lfloor \left(\frac{1}{cw}\right)^{1/2}\left(\frac{1}{\widehat{\lambda}_{T_2}}\right)^{(k+1)/2} + \epsilon \right\rfloor + 1\right)\right\},$$

and

$$II = \left(\frac{1}{cw}\right)^{1/2} E\left\{\left(\frac{1}{\widehat{\lambda}_{T_2}}\right)^{(k+1)/2}\right\} + \epsilon + E(\psi_{T_2}). \tag{A.17}$$

It follows from Hall (1983) that, as $c \to 0$, $I = o(1)$ and ψ_{T_2} is uniform over (0,1). Thus, we obtain from (A.16) and (A.17) that,

$$E(N_2) = \left(\frac{1}{cw}\right)^{1/2} E\left\{\left(\frac{1}{\widehat{\lambda}_{T_2}}\right)^{(k+1)/2}\right\} + \frac{1}{2} + \epsilon + o(1). \tag{A.18}$$

Now we first evaluate $E\left\{\left(1/\widehat{\lambda}_{T_2}\right)^{(k+1)/2}\right\}$. To this end, under the condition (3.10), we can write (3.7) as,

$$L_2 = \inf\Bigg\{n \geq 0 : q(m^*+n)(q(m^*+n)+1) \geq \eta\left(\frac{1}{cw}\right)^{1/2}\left(\frac{1}{\lambda}\right)^{(k+1)/2} \times q(m^*+n)\left(\frac{\lambda}{\widehat{\lambda}_{q(m^*+n)+1}}\right)^{(k+1)/2}\Bigg\},$$

which can further be written as,

$$L_2 = \inf\left\{n \geq m^* : qn(pn+1) \geq \eta n^*(qn)\left(\frac{\lambda}{\widehat{\lambda}_{qn+1}}\right)^{(k+1)/2}\right\}. \tag{A.19}$$

Further, we have,

$$\frac{qn\lambda}{\widehat{\lambda}_{qn+1}} \sim \chi^2_{qn}. \tag{A.20}$$

(A.20) can be expressed as the summation of 'n' *iid* chi-squared random variables with 'q' degrees of freedom, i.e., $\sum_{j=1}^{n} Z_j$, where $Z_j, j = 1, 2, ..., n$ are *iid* χ^2_q random variables.

Using (A.20), we rewrite (A.19) as,

$$L_2^* = \inf\left\{n \geq m^* : \sum_{j=1}^{n} Z_j \leq \frac{(nq)^{(k+3)/(k+1)}}{(\eta n^*)^{2/(k+1)}}\left(1 + \frac{1}{nq}\right)^{2/(k+1)}\right\}, \tag{A.21}$$

where the stopping time $L_2^* = L_2 + m^*$.

Comparing (A.21) with equation (1.1) of Woodroofe (1977), we obtain in his notations:

$$S_n = \sum_{j=1}^{n} Z_j, c = \frac{q^{(k+3)/(k+1)}}{(\eta n^*)^{2/(k+1)}}, \alpha = \frac{k+3}{k+1}, \beta = \frac{k+1}{2}, \lambda = \frac{\eta n^*}{q},$$

$$L_n = \left\{1 + \frac{1}{nq}\right\}^{2/(k+1)}, L_0 = \frac{2}{q(k+1)}, \mu = q, \tau^2 = 2q.$$

It now follows from his Theorem 2.4 that, for all $m > (k+1)/2q$, as $c \to 0$,

$$E(L_2^*) = (\eta n^*)q^{-1} + \left(\frac{k+1}{2}\right)\nu q^{-1} - \left\{1 + \frac{(k+1)(k+3)}{4}\right\} q^{-1} + o(1). \tag{A.22}$$

provided that $m^* > 2/q$, where 'ν' is specified. By noting that, $T_2 = qL_2^* + 1$ with probability 1, it follows from (A.22) that,

$$E(T_2) = \eta n^* + \left(\frac{k+1}{2}\right)\nu - \frac{(k+1)(k+3)}{4} + o(1). \tag{A.23}$$

Now, let us consider the difference,

$$D_a = T_2 - \eta \left(\frac{1}{cw}\right)^{1/2} \left(\frac{1}{\widehat{\lambda}_{T_2}}\right)^{(k+1)/2}. \tag{A.24}$$

It follows from Woodroofe (1977) that, mean of the asymptotic distribution of D_a is $(k+1)\nu/2$. Now from (A.23) and (A.24), we have,

$$\left(\frac{1}{cw}\right)^{1/2} E\left\{\left(\frac{1}{\widehat{\lambda}_{T_2}}\right)^{(k+1)/2}\right\} = n^* - \frac{(k+1)(k+3)}{4\eta} + o(1). \tag{A.25}$$

Now (3.11) follows by putting (A.25) in (A.18). In order to obtain (3.12), we proceed as follows:

The risk function associated with our stopping rule 'N_2' is given by:

$$R_{N_2}(c) = w\left\{E\left(\frac{n^*}{N_2}\right)^2\right\}. \tag{A.26}$$

Thus, we have to obtain the expression for $E(n^*/N_2)^2$. Applying Taylor's series expansion, we get:

$$E\left(\frac{n^*}{N_2}\right)^2 = 1 - \frac{2}{n^*}E(N_2 - n^*) + \frac{3}{n^{*2}}E(N_2 - n^*)^2 U^{-4}, \tag{A.27}$$

where, $\mid U - 1 \mid \leq \mid (N_2/n^*) - 1 \mid$. Moreover, we have from Ghosh and Mukhopadhyay (1975),

$$\frac{N_2 - n^*}{n^{*1/2}} \longrightarrow N\left(0, \frac{(k+1)^2}{2q}\right). \tag{A.28}$$

and 'U' is uniformly integrable [See Lemma 3.4 in Chaturvedi et al. (2019)]. Now using (3.11) and (A.28), we have from (A.27),

$$E\left(\frac{n^*}{N_2}\right)^2 = 1 - \frac{1}{n^*}\left[2\left(\epsilon + \frac{1}{2}\right) - \frac{(k+1)(k+3)}{2\eta} - \frac{3(k+1)^2}{2q}\right] + o(c^{-1/2}). \tag{A.29}$$

(3.12) now follows by putting (A.29) in (A.26). Proof of Theorem 3.2 is now complete.