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Generalization of the Newsvendor Problem with gamma demand distribution by Asymmetric Losses

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Generalization of the Newsvendor Problem with gamma demand distribution by Asymmetric Losses

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Abstract

Classical newsboy problem has been extended in many directions to accommodate more realistic inventory scenarios. In this paper we consider single period inventory problem where the product is short-lived and the severity of leftover and shortages are not same. Such a model would play important role in deciding optimum inventory level of, e.g. greengrocers or supermarkets, among others who are selling such short-lived items. In this work we introduce the concept of importance function. Further, we characterize it in terms of the dimension of the cost function and relation with realized demand and inventory level. The model we propose considers different importance for leftover and shortage. We provide the conditions for existence of feasible solutions of the optimal order quantity determination problem. We also present results from a number of numerical instances with specific importance functions. The numerical results show that higher importance to a type of loss results in conservative inventory orders in the direction of the corresponding importance.

1 Introduction

Let us consider the classical newsvendor's problem, who has to decide the order quantity at the beginning of the period, after which the remaining inventory expires. With stochastic demand, the problem for the newsvendor

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is to determine the optimal order quantity so that it balances the losses due to shortage and leftover. This problem has drawn considerable amount of attention over past hundred years since its introduction (Harris, 1913) due to its applicability in analogous problems related to inventory management, revenue management, supply chain coordination and many others.

In this paper we propose a generalization of the newsvendor problem with different degree of sensitivity towards leftover and shortage. Classical newsvendor problem treats these two losses with equal importance and measures the corresponding cost in proportion to the difference between demand and inventory level with unit importance. However, in many situations leftover and shortage would warrant different importance. For example, supermarkets with perishable commodities would face higher customer churn in case of shortage (Fitzsimons, 2000), whereas financially constrained vendors like greengrocer or newsboy would find leftover more severe (Dada & Hu, 2008).

Here we consider the case of a newsvendor, who puts more importance on leftover than shortage (to be called poor newsvendor henceforth) in absence of significant salvation cost. Hence, the loss incurred due to leftover inventory should be more severe than merely the quantity of unsold inventory. Indeed, the severity of leftover would increase with depleting demand. Intuitively, here the poor newsvendor would end up ordering less quantity than the optimum level in a classical newsvendor problem (Dada & Hu, 2008). Conversely, a newsvendor with more sensitivity towards shortage than leftover (to be called rich newsvendor hereafter), would find severity of shortage increasing with demand. As a result, the rich newsvendor will be expected to order more than the optimal order quantity in classical case (Dana Jr & Petruzzi, 2001, p.1495). Here we propose a cost setup where the sensitivity of the newsvendor to leftover and shortage is expressed through two different dimensionless importance functions.

Deviation from classical newsvendor based optimal order quantity has been discussed in the literature majorly from two angles. Bounded rationality theory suggests that newsvendor are prone to subjective biases in deciding optimum order quantity (Su, 2008). Schweitzer & Cachon (2000) and Vipin & Amit (2019), among others, explain the decision bias of the newsvendor, through experiments, with high (or low) profit margin of products. On the other hand, Eeckhoudt et al. (1995); Wang et al. (2009) studied the newsvendor problem based on risk-appetite of the newsvendor. It seems no work so far has considered explicitly the biases due to different sensitivities of the newsvendor to shortage and leftover losses.

As argued in Kahneman & Tversky (1979) and Nagarajan & Shechter (2013), that perceived disutility of the loss is higher than the perceived utility of the equivalent gain, we posit the sensitivity to the types losses as the perceived disutility, which is higher for leftover (shortage) loss than shortage (leftover) loss for the poor (rich) newsvendor. Further, the perceived disutility will increase with amount of leftovers (shortages) for poor (rich) newsvendor. Hence, we model the disutilities of the newsvendor through two dimensionless non-linear importance functions in addition to the classical newsvendor setup.

Although some works report non-linearity in the cost function, asymmetric importance of the two remains unaddressed to the best of knowledge of the authors. For example, Gerchak & Wang (1997) proposed a power type shortage cost for liquid asset allocation problem. However, this work does not consider the leftover cost and the shortage cost has same power of currency as dimension. Kyparisis & Koulamas (2018) studied the single period price-setting newsvendor problem with both salvage revenue and shortage cost as a quadratic function. Rosling (2002) studied periodic review inventory system with non-linear shortage cost, where non-linearity appears through the cumulative distribution function (CDF) of the total demand. Some other types cost of functions, which has been modeled with non-linearity includes, among many others, piecewise quadratic holding cost by Parlar & Rempala (1992a,b), exponential holding cost by Pal et al. (2015). Halman et al. (2012) studied the newsvendor problem with non-decreasing piecewise linear procurement cost. Inventory system with nonlinear stock dependent demand and holding cost has been studied more recently by Cárdenas-Barrón et al. (2018) (see the reference therein).

Another important aspect of the newsvendor problem is the order of dimension of cost function. In classical newsvendor problem, the dimension of the total cost is in currency unit (say \$) (Rosling, 2002). However, power type costs proposed so far in the literature does not maintain uniformity in the dimensions. In this work, we overcome this problem by constructing unit-free non-linear importance functions. In turn this approach also ensures that shortage (leftover) penalty is proportional to the shortage (leftover) quantity (Kyparisis & Koulamas, 2018, p.65). Further, the non-linearity of the proposed importance functions results in decrease (increase) in cost due to shortage (leftover) with increasing order quantity and the reverse with increasing realized demand.

Rest of the paper is arranged as follows. Section 2 details the mathematical model of the newsvendor problem with non-linear leftover and shortage importance functions. Numerical examples are given in Section 3. We conclude the paper in Section 4 with a brief discussion on our findings.

2 Mathematical Model

In this section, we describe the mathematical model for the newsvendor problem with non-linear leftover and shortage importance functions. Before starting into mathematical model, first we list down the nomenclatures in table 1.

Table 1: Nomenclature List

Symbol	Explanation
EC	Expected cost of shortages and leftover of products
c_e	Leftover cost
c_s	Shortage cost
q	Order quantity (Decision variable)
$f(x)$	Probability density function of demand
$F(x)$	Comulative distribution function of demand
m	Coefficient of importance for leftover
n	Coefficient of importance for shortage

The importance functions I_1 and I_2 are added in the newsvendor cost formula to provide the weightage to leftover and shortage costs;

$$C = \begin{cases} c_e(q-x)I_1, & if\ x \leq q \\ c_s(x-q)I_2, & if\ x > q \end{cases} \tag{1}$$

The importance functions I_1 and I_2 should be so selected that for realized $x \leq q$, it will decrease with demand x but increase with order quantity q and vice-versa for $x > q$. Such two functions are $L_1(q) = \left(\frac{q}{x}\right)^m$ and $L_2(q) = \left(\frac{x}{q}\right)^n$ (See appendix A.1 for more details).

$$C = \begin{cases} c_e(q-x)\left(\dfrac{q}{x}\right)^m, & if\ x \leq q \\ c_s(x-q)\left(\dfrac{x}{q}\right)^n, & if\ x > q \end{cases} \quad \forall \quad m,n \geq 0;\ integer \tag{2}$$

The expected cost function then can be written as,

$$EC = c_e \int_0^q (q-x)\left(\frac{q}{x}\right)^m f(x)dx + c_s \int_q^\infty (x-q)\left(\frac{x}{q}\right)^n f(x)dx \qquad (3)$$

To find the optimum order quantity q for which the expected cost is minimum, we differentiate the expected cost EC with respect to order quantity q, (Refer appendix A.2 for derivation).

$$\frac{d}{dq}(EC) = 0$$

$$\Rightarrow c_e \int_0^q \left[\left(\frac{q}{x}\right)^m + m\frac{(q-x)}{x}\left(\frac{q}{x}\right)^{m-1}\right] f(x)dx$$
$$- c_s \int_q^\infty \left[\left(\frac{x}{q}\right)^n + nx\frac{(x-q)}{q^2}\left(\frac{x}{q}\right)^{n-1}\right] f(x)dx = 0 \qquad (4)$$

2.1 Uniform Distribution

For simplicity, let's consider demand follows Uniform distribution with parameter A and B, where $0 < A < B \leq \infty$.

$$EC = c_e \int_A^q (q-x)\left(\frac{q}{x}\right)^m \left(\frac{1}{B-A}\right)dx + c_s \int_q^B (x-q)\left(\frac{x}{q}\right)^n \left(\frac{1}{B-A}\right)dx \qquad (5)$$

To find the optimum order quantity q for which the expected cost is minimum, we differentiate the expected cost EC with respect to order quantity q (Refer equation 4),

$$\frac{d}{dq}(EC) = 0$$

$$c_e \int_A^q \left[\left(\frac{q}{x}\right)^m + m\frac{(q-x)}{x}\left(\frac{q}{x}\right)^{m}\right]\left(\frac{1}{B-A}\right)dx$$
$$- c_s \int_q^B \left[\left(\frac{x}{q}\right)^n + nx\frac{(x-q)}{q^2}\left(\frac{x}{q}\right)^{n-1}\right]\left(\frac{1}{B-A}\right)dx = 0 \qquad (6)$$

Since general solutions for the case $m = 1$ and $m = 2$, for any n is not tractable, we solve it in three different cases.

Proposition 1. *The optimum order quantity q^* for the newsvendor problem is obtained by solving following equations;*

1. $$\left(\frac{c_e}{B-A}\right)\left[\frac{(m+1)\left(A^{-m+1}q^m-q\right)}{m-1}-\frac{m\left(A^{-m+2}q^{m-1}-q\right)}{m-2}\right] + \left(\frac{c_s}{B-A}\right)\left[\frac{(n-1)\left(B^{n+1}q^{-n}-q\right)}{n+1}-\frac{n\left(B^{n+2}q^{-n-1}-q\right)}{n+2}\right]=0$$
   for m = m - {1,2} and n = n;

2. $$\left(\frac{c_e}{B-A}\right)[2q\ln q-2q\ln A-q+A] + \left(\frac{c_s}{B-A}\right)\left[\frac{(n-1)\left(B^{n+1}q^{-n}-q\right)}{n+1}-\frac{n\left(B^{n+2}q^{-n-1}-q\right)}{n+2}\right]=0$$
   for m = 1 and n = n;

3. $$\left(\frac{c_e}{B-A}\right)\left[2q\ln A-2q\ln q-3q+\frac{3q^2}{A}\right] + \left(\frac{c_s}{B-A}\right)\left[\frac{(n-1)\left(B^{n+1}q^{-n}-q\right)}{n+1}-\frac{n\left(B^{n+2}q^{-n-1}-q\right)}{n+2}\right]=0$$
   for m = 2 and n = n;

Proof: See appendix B.1.

Notice that the first order condition requires solution of a polynomial equation. The following proposition provides the conditions for existence of the roots.

Proposition 2.

1. *For m = 0 and $n \geq 1$; unique real positive root of the FOC equation exist*
2. *For $m \geq 1$ and n = 0; either there are even number of positive real roots of the FOC equation or the problem is infeasible*
3. *For $m \geq 3$ and $n \geq 1$, either one or three positive real roots of the FOC equation exist.*

Proof: Following Descartes sign rule the proof is immediate.

Proposition 3. *The expected cost function is unimodal with respect to the optimum order quantity q^* for all the three cases.*

Proof: See appendix B.2.

2.2 Gamma Distribution

Suppose demand follows gamma distribution with shape parameter α and scale parameter θ. The probability density function can be written as,

$$f(x;\alpha,\theta)=\frac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\theta^{\alpha}\Gamma(\alpha)},\ x>0;\ \alpha,\theta>0$$

The expected cost function is written as,

$$\begin{aligned}
EC &= c_e\int_0^q (q-x)\left(\frac{q}{x}\right)^m\left(\frac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\theta^{\alpha}\Gamma(\alpha)}\right)dx + c_s\int_q^\infty (x-q)\left(\frac{x}{q}\right)^n\left(\frac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\theta^{\alpha}\Gamma(\alpha)}\right)dx \\
&= \frac{1}{\Gamma(\alpha)}\left[c_e\frac{q^m}{\theta^m}\{q\gamma(\alpha-m,q)-\gamma(\alpha-m+1;q)\}\right. \\
&\qquad \left.+\,c_s\left(\frac{\theta}{q}\right)^n\{\theta\Gamma(\alpha+n+1,q)-q\Gamma(\alpha+n;q)\}\right] \\
&= \frac{1}{\Gamma(\alpha)}\left[c_e\frac{q^m}{\theta^m}\left\{\gamma(\alpha-m,q)(q-\alpha+m)+q^{\alpha-m}e^{-q}\right\}+c_s\left(\frac{\theta}{q}\right)^n\Gamma(\alpha+n,q)\right. \\
&\qquad \left.\times\left\{(\theta(\alpha+n)-q)+q^{\alpha+n}e^{-q}\right\}\right] \qquad (7)
\end{aligned}$$

where, $\gamma(s,q)=\int_0^q u^{s-1}e^{-u}du$ is the lower gamma function and $\Gamma(s,q)=\int_q^\infty u^{s-1}e^{-u}du$ is the upper gamma function.

In the following section, we present numerical instances and corresponding optimum order quantity for uniform and gamma distributed demand.

3 Numerical Analysis

Let us suppose the demand follows $U(50,100)$. Further, we set $c_e=3$, $c_s=3$ and vary the importance parameters m and n from 0 to 8. In the following table we present the optimum order quantity q^* for each pair of m and n.

Table 2: Effect of m and n on EOQ for Uniform distribution

Value of n	Value of m								
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
0	75.00	71.07	68.10	65.85	64.12	62.75	61.64	60.72	59.96
1	78.28	74.58	71.55	69.13	67.18	65.60	64.29	63.19	62.26
2	80.64	77.20	74.24	71.78	69.73	68.02	66.59	65.37	64.32
3	82.43	79.24	76.41	73.97	71.89	70.11	68.60	67.29	66.16
4	83.85	80.90	78.19	75.81	73.73	71.93	70.37	69.01	67.81
5	85.01	82.25	79.68	77.37	75.32	73.52	71.94	70.54	69.31
6	85.98	83.40	80.96	78.73	76.72	74.93	73.34	71.93	70.67
7	86.81	84.40	82.06	79.91	77.95	76.19	74.60	73.18	71.90
8	87.53	85.24	83.03	80.95	79.05	77.31	75.74	74.32	73.04

Similarly, we conducted numerical analysis for gamma distribution with shape parameter α and scale parameter θ with value 75 and 1.0 respectively. The other parameters are similar with uniform distribution.

Table 3: Effect of m and n on EOQ for Gamma distribution

Value of n	Value of m								
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
0	74.65	73.70	72.80	71.95	71.15	70.35	69.65	68.95	68.25
1	75.60	74.65	73.75	72.90	72.10	71.35	70.60	69.90	69.20
2	76.45	75.50	74.60	73.80	73.00	72.20	71.50	70.80	70.10
3	77.25	76.35	75.45	74.60	73.80	73.05	72.30	71.60	70.90
4	78.05	77.10	76.20	75.40	74.60	73.80	73.10	72.35	71.70
5	78.75	77.85	76.95	76.10	75.30	74.55	73.80	73.10	72.40
6	79.45	78.50	77.65	76.80	76.00	75.25	74.50	73.80	73.10
7	80.10	79.15	78.30	77.45	76.65	75.90	75.15	74.45	73.75
8	80.70	79.80	78.90	78.10	77.30	76.50	75.80	75.10	74.40

Notice, the optimum order quantity is decreasing in m for fixed n and increasing with n for fixed m. However, for $m = n$ the optimal order quantity decreases with the coefficient of importance. The intuition behind the result is as follows. As soon as the importance of leftover loss increases for a given level of shortage importance, the newsvendor orders less to minimize the most important loss, i.e. the leftover loss. Similarly, the reverse pattern is observed in case of increasing shortage importance for a given level of leftover importance. For $m = n$, the decreasing values of Q^* for increasing importance could be interpreted as the more conservative ordering pattern or in other words growing risk-averseness of the newsvendor. The above results are also in-line with Pranoto (2005, proposition 2 on p.21), i.e. optimal order quantity of a poor loss-averse newsvendor is less than the optimal order quantity of rich loss-averse newsvendor.

4 Conclusion

This paper proposes a generalization of the newsvendor problem with different degree of severity towards leftovers and shortages. We propose two novel dimensionless importance functions for different levels of severity of the leftover and shortage losses and in turn, it explains the decision biases of newsvendor based on financial wealth (i.e. poor or rich). Further we have shown that for uniform and gamma demand, existence of the optimal order quantity is dependent on the degree of importance and provided the conditions for the same. Numerical examples show that poor newsvendor is more concerned for leftover than shortages and orders less whereas, a rich newsvendor is more concerned about the shortages than leftovers resulting in higher optimum order quantity. We are considering more general class of demand distribution as extensions of this work.

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Appendix A

A.1

For $x \leq q$, importance function $(q-x)\left(\frac{q}{x}\right)^m$ is increase with q and decrease with x. We get,

$$\begin{aligned}\frac{d}{dq}\left[(q-x)\left(\frac{q}{x}\right)^m\right] &= \left(\frac{q}{x}\right)^m + (q-x)m\left(\frac{q}{x}\right)^{m-1}\left(\frac{1}{x}\right)\\ &= \left(\frac{q}{x}\right)^m + m\left(\frac{q-x}{x}\right)\left(\frac{q}{x}\right)^{m-1}\\ &> 0, \qquad \forall x\&q \geq 0 \end{aligned} \tag{A.1}$$

$$\begin{aligned}\frac{d}{dx}\left[(q-x)\left(\frac{q}{x}\right)^m\right] &= -\left(\frac{q}{x}\right)^m + (q-x)m\left(\frac{q}{x}\right)^{m-1}\left(\frac{-1}{x^2}\right)\\ &= -\left(\frac{q}{x}\right)^m - m\left(\frac{q-x}{x^2}\right)\left(\frac{q}{x}\right)^{m-1}\\ &\leq 0, \qquad \forall x\&q \geq 0 \end{aligned} \tag{A.2}$$

For $x > q$, importance function $(x-q)\left(\frac{x}{q}\right)^m$ is increase with x and decrease with q. We get,

$$\begin{aligned}\frac{d}{dq}\left[(x-q)\left(\frac{x}{q}\right)^m\right] &= -\left(\frac{x}{q}\right)^n + (x-q)n\left(\frac{x}{q}\right)^{n-1}\left(\frac{-x}{q^2}\right)\\ &= -\left(\frac{x}{q}\right)^n - nx\left(\frac{x-q}{q^2}\right)\left(\frac{x}{q}\right)^{n-1}\\ &\leq 0, \qquad \forall x\&q \geq 0 \end{aligned} \tag{A.3}$$

$$
\begin{aligned}
\frac{d}{dx}\left[(x-q)\left(\frac{x}{q}\right)^{m}\right] &= \left(\frac{x}{q}\right)^{n} + (x-q)n\left(\frac{x}{q}\right)^{n-1}\left(\frac{1}{q}\right) \\
&= \left(\frac{x}{q}\right)^{n} + n\left(\frac{x-q}{q}\right)\left(\frac{x}{q}\right)^{n-1} \\
&> 0, \qquad \forall x \& q \geq 0
\end{aligned}
\tag{A.4}
$$

A.2

Differentiate Expected cost function EC with order quantity q to find the optimum order quantity,

$$
\frac{d}{dq}(EC) = 0
$$

$$
\begin{aligned}
&\frac{d}{dq}\left(c_e \int_0^q (q-x)\left(\frac{q}{x}\right)^m f(x)dx + c_s \int_q^\infty (x-q)\left(\frac{x}{q}\right)^n f(x)dx\right) = 0 \\
&c_e \int_0^q \frac{\partial}{\partial q}\left[(q-x)\left(\frac{q}{x}\right)^m f(x)\right]dx + c_s \int_q^\infty \frac{\partial}{\partial q}\left[(x-q)\left(\frac{x}{q}\right)^n f(x)\right]dx = 0 \\
&c_e \int_0^q \left[\left(\frac{q}{x}\right)^m + m\frac{(q-x)}{x}\left(\frac{q}{x}\right)^{m-1}\right]f(x)dx - c_s \int_q^\infty \left[\left(\frac{x}{q}\right)^n + nx\frac{(q-x)}{q^2}\left(\frac{x}{q}\right)^{n-1}\right]f(x)dx = 0
\end{aligned}
\tag{A.5}
$$

Appendix B

B.1

For general $m = m, n = n$;

$$c_e \int_A^q \left[\left(\frac{q}{x}\right)^m + m\frac{(q-x)}{x}\left(\frac{q}{x}\right)^{m-1}\right]\left(\frac{1}{B-A}\right)dx$$
$$- c_s \int_q^B \left[\left(\frac{x}{q}\right)^n + nx\frac{(q-x)}{q^2}\left(\frac{x}{q}\right)^{n-1}\right]\left(\frac{1}{B-A}\right)dx = 0$$
$$\left(\frac{c_e}{B-A}\right)\left[\frac{(m+1)\left(A^{-m+1}q^m - q\right)}{m-1} - \frac{m\left(A^{-m+2}q^{m-1} - q\right)}{m-2}\right]$$
$$+ \left(\frac{c_s}{B-A}\right)\left[\frac{(n-1)\left(B^{n+1}q^{-n} - q\right)}{n+1} - \frac{n\left(B^{n+2}q^{-n-1} - q\right)}{n+2}\right] = 0 \tag{B.1}$$

For general $m = 1, n = n$;

$$\frac{d}{dq}\left[c_e \int_A^q (q-x)\left(\frac{q}{x}\right)^1 f(x)\left(\frac{1}{B-A}\right)\right]$$
$$- c_s \int_q^B \left[\left(\frac{x}{q}\right)^n + nx\frac{(q-x)}{q^2}\left(\frac{x}{q}\right)^{n-1}\right]\left(\frac{1}{B-A}\right)dx = 0$$
$$\left(\frac{c_e}{B-A}\right)[2q\ln q - 2q\ln A - q + A]$$
$$+ \left(\frac{c_s}{B-A}\right)\left[\frac{(n-1)\left(B^{n+1}q^{-n} - q\right)}{n+1} - \frac{n\left(B^{n+2}q^{-n-1} - q\right)}{n+2}\right] = 0 \tag{B.2}$$

For general $m = 2, n = n$;

$$\frac{d}{dq}\left[c_e \int_A^q (q-x)\left(\frac{q}{x}\right)^2 f(x)\left(\frac{1}{B-A}\right)\right]$$
$$- c_s \int_q^B \left[\left(\frac{x}{q}\right)^n + nx\frac{(q-x)}{q^2}\left(\frac{x}{q}\right)^{n-1}\right]\left(\frac{1}{B-A}\right)dx = 0$$
$$\left(\frac{c_e}{B-A}\right)\left[2q\ln A - 2q\ln q - 3q + \frac{3q^2}{A}\right]$$
$$+ \left(\frac{c_s}{B-A}\right)\left[\frac{(n-1)\left(B^{n+1}q^{-n} - q\right)}{n+1} - \frac{n\left(B^{n+2}q^{-n-1} - q\right)}{n+2}\right] = 0 \tag{B.3}$$

B.2

In present appendix we proved the unimodality of the optimum order quantity, q^* for all the three cases of uniform distribution,

For general $m = m, n = n$;

$$
\begin{aligned}
\frac{d^2}{dq^2}(EC) &= \frac{d}{dq}\left(\frac{d}{dq}(EC)\right) \\
&= \frac{d}{dq}\left[\left(\frac{c_e}{B-A}\right)\left[\frac{(m+1)\left(A^{-m+1}q^{m}-q\right)}{m-1} - \frac{m\left(A^{-m+2}q^{m-1}-q\right)}{m-2}\right]\right] \\
&\qquad + \frac{d}{dq}\left[\left(\frac{c_s}{B-A}\right)\left[\frac{(n-1)\left(B^{n+1}q^{-n}-q\right)}{n+1} - \frac{n\left(B^{n+2}q^{-n-1}-q\right)}{n+2}\right]\right] \\
&= \left(\frac{c_e}{B-A}\right)\left[\frac{(m+1)\left((m)A^{-m+1}q^{m-1}-1\right)}{m-1} - \frac{m\left((m-1)A^{-m+2}q^{m-2}-1\right)}{m-2}\right] \\
&\qquad + \left(\frac{c_s}{B-A}\right)\left[\frac{(n-1)\left((-n)B^{n+1}q^{-n-1}-1\right)}{n+1} - \frac{n\left((-n-1)B^{n+2}q^{-n-2}-1\right)}{n+2}\right] \\
&= \left(\frac{c_e}{B-A}\right)\left[\frac{2}{(m-1)(m-2)} + m\left(\frac{q}{A}\right)^{m-2}\left(\frac{m+1}{m-1}\left(\frac{q}{A}\right) - \frac{m-1}{m-2}\right)\right] \\
&\qquad + \left(\frac{c_s}{B-A}\right)\left[\frac{2}{(n+1)(n+2)} + n\left(\frac{B}{q}\right)^{n+1}\left(\frac{n+1}{n+2}\left(\frac{B}{q}\right) - \frac{n-1}{n+1}\right)\right] \\
&\geq 0, \qquad \forall\, m = m - \{1,2\} \text{ and } n = n \qquad\qquad \text{(B.4)}
\end{aligned}
$$

For general $m = 1, n = n$;

$$
\begin{aligned}
\frac{d^2}{dq^2}(EC) &= \frac{d}{dq}\left(\frac{d}{dq}(EC)\right)\\
&= \frac{d}{dq}\left[\left(\frac{c_e}{B-A}\right)[2q\ln q - 2q\ln A - q + A]\right]\\
&\qquad + \frac{d}{dq}\left[\left(\frac{c_s}{B-A}\right)\left[\frac{(n-1)\left(B^{n+1}q^{-n}-q\right)}{n+1} - \frac{n\left(B^{n+2}q^{-n-1}-q\right)}{n+2}\right]\right]\\
&= \left(\frac{c_e}{B-A}\right)[2\ln q - 2\ln A + 1]\\
&\qquad + \left(\frac{c_s}{B-A}\right)\left[\frac{(n-1)\left((-n)B^{n+1}q^{-n-1}-1\right)}{n+1} - \frac{n\left((-n-1)B^{n+2}q^{-n-2}-1\right)}{n+2}\right]\\
&= \left(\frac{c_e}{B-A}\right)[2\ln q - 2\ln A + 1]\\
&\qquad + \left(\frac{c_s}{B-A}\right)\left[\frac{2}{(n+1)(n+2)} + n\left(\frac{B}{q}\right)^{n+1}\left(\frac{n+1}{n+2}\left(\frac{B}{q}\right) - \frac{n-1}{n+1}\right)\right]\\
&\geq 0, \qquad\qquad \forall\, m = 1 \text{ and } n = n
\end{aligned}
\tag{B.5}
$$

For general $m = 2, n = n$;

$$
\begin{aligned}
\frac{d^2}{dq^2}(EC) &= \frac{d}{dq}\left(\frac{d}{dq}(EC)\right)\\
&= \frac{d}{dq}\left[\left(\frac{c_e}{B-A}\right)\left[2q\ln A - 2q\ln q - 3q + \frac{3q^2}{A}\right]\right]\\
&\qquad + \frac{d}{dq}\left[\left(\frac{c_s}{B-A}\right)\left[\frac{(n-1)\left(B^{n+1}q^{-n}-q\right)}{n+1} - \frac{n\left(B^{n+2}q^{-n-1}-q\right)}{n+2}\right]\right]\\
&= \left(\frac{c_e}{B-A}\right)\left[2\ln A - 2\ln q - 5 + \frac{6q}{A}\right]\\
&\qquad + \left(\frac{c_s}{B-A}\right)\left[\frac{(n-1)\left((-n)B^{n+1}q^{-n-1}-1\right)}{n+1} - \frac{n\left((-n-1)B^{n+2}q^{-n-2}-1\right)}{n+2}\right]\\
&= \left(\frac{c_e}{B-A}\right)\left[2\ln A - 2\ln q - 5 + \frac{6q}{A}\right]\\
&\qquad + \left(\frac{c_s}{B-A}\right)\left[\frac{2}{(n+1)(n+2)} + n\left(\frac{B}{q}\right)^{n+1}\left(\frac{n+1}{n+2}\left(\frac{B}{q}\right) - \frac{n-1}{n+1}\right)\right]\\
&\geq 0, \qquad\qquad \forall\, m = 2 \text{ and } n = n
\end{aligned}
\tag{B.6}
$$